On the possibility of classical client secure delegated computation

Alexandru Cojocaru

S1575638

Master of Science

Artificial Intelligence

School of Informatics

University of Edinburgh

2016

Abstract

As quantum technologies are showing a growing progress, we will be soon facing a physical implementation of a quantum computer. It is well-known the fact that the quantum model can surpass the computational efficiency of any classical machine. Therefore, we consider the problem of delegating a hard computation to a quantum computer, such that the input of the computation remains private.

We analyse from a complexity theoretic point of view what classes of problems can be solved in this encryption scheme. In this way, we determine to what extent we can securely take advantage of the power of a quantum server.

We indicate that there is a strong relation between how hard is to encrypt a problem and how hard is to solve it. For this reason, we further investigate what classes of problems admit this type of encryption.

Moreover, we examine a quantum protocol which also performs this secure delegated task and which achieves additional security properties. Finally, we inspect the possibility of creating a classical encryption scheme which inherits the security conditions imposed by this quantum protocol.

Declaration

I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification except as specified.

(*Alexandru Cojocaru*
s1575638)

Table of Contents

1. Introduction

The evolution of traditional computers as described by Moore's law is challenged by a series of technological limitations. Since the power of these machines is proportional to the number of transistors they have, the trend has been the construction of smaller and more efficient components. For this reason, we will be soon facing computer constituents of size no bigger than an atom [1]. The main issue that emerges is that at this scale we cannot use anymore the laws of classical physics to describe the behaviour of these devices. Instead, we need to use the theories which govern the microscopic world [2, 3]. Thus, at this point quantum computing steps in.

Quantum Computing arose from the idea of modelling any natural phenomena using the laws of physics. The quantum mechanics principles allow us to describe any process of the nature [1, 4]. Therefore, in order to develop more powerful computing devices, scientists decided to harness quantum properties of atoms, which would provide the basis for the memory and processor of the new machine [5].

Bernstein and Vazirani [4, 5] were the first to reach the remarkable conclusions that quantum computers can solve certain problems much faster than the best classical computational model, the probabilistic Turing Machine and that no possible evolution of classical computers could compare to the power brought by the quantum paradigm [6]. This model of computation based on quantum mechanics has first attained outstanding practical results when Peter Shor and Lov Grover developed quantum algorithms for problems such as searching an element in an unsorted database or factoring large numbers [7], which outclassed the efficiency of the best known classical solutions.

The advantages of quantum computing were also emphasized by Isaac Chuang and Michael Nielsen [1] who grappled with the question: "Why this struggle to obtain full control of quantum systems?". They argue that the main reason lies in the fact that the use of quantum theory for information processing tasks leads to an impressive speedup, which is out of reach for standard traditional machines [1, 3]. Moreover, even if this outstanding improvement in speed would not be applicable for any type of problem, the breakthrough achieved through exploiting quantum processes would allow us to gain a better understanding of the Universe [3, 5] and even uncover unknown physical phenomena. In this sense, Seth Lloyd has developed a quantum algorithm which can simulate certain quantum physical systems [8] and which could therefore play an important role in areas such as high energy physics, chemistry or cosmology [6, 8].

However, perhaps the most significant impact brought by quantum computing is revealed in the field of cryptography. Firstly, current classical public key cryptography is based on computationally infeasible problems [9]. This fact illustrates their vulnerability against a quantum computer. Because a quantum computer is able to efficiently solve hard problems such as factorization, or discrete logarithm [4, 10], it could easily break the most popular cryptographic algorithms: RSA, Diffie-Hellman, ElGamal, etc.

On the other hand, quantum information theory also provides some very promising solutions for cryptographic tasks [11]. The most trivial example is the ability to generate pure random numbers. This would guarantee that we can construct keys which secure the privacy of data, in such a way that it would be impossible for an attacker to estimate their values [3, 12]. From this property was born the concept of quantum cryptography [11, 12]. Specifically, quantum cryptography refers to a key distribution protocol which provides information theoretic security. This means that they would not only be secure against both classical and quantum adversaries, but against any adversaries. The reason is that the security is based on the fundamental theorems of quantum physics [13] instead of difficult mathematical computations. To be precise, the keys are generated by measuring certain properties of quantum particles and by using the "Uncertainty Principle" [12] we can derive that any malicious eavesdropper is going to be detected.

Our project is related to an important component of cryptography, namely *encryption,* and in fact, we will be focusing on the following problem:

Suppose a client wants to determine the result of a computation on some input, but lacks the necessary resources to perform the computation himself. He has access to a server which has unbounded power and can solve the problem and send the outcome back to the client. However, the main issue is that the client wants to ensure the privacy of his data, so he needs to find a way to hide the input of the computation from the server. Moreover, as we assumed the server has limitless computational capabilities, the encryption process must not be based on solving hard mathematical problems, because then, the server could crack the cipher and get the plaintext input.

The problem described is known as *secure delegated computation* [14]. Considering that an encryption solution to this problem is meant to be information-theoretically secure [9] (not even an unbounded powerful attacker could decipher it), this type of protocol would play a decisive role in the advancement of cloud computing. Additionally, the invulnerability assertion makes this model of encryption a perfect candidate for a cryptographic primitive of a strong, reliable cryptosystem [4, 9].

Now, taking into account the prospective quantum speedup in solving hard problems, we analyse the secure delegated computation task in the situation where we use a quantum computer in the role of the powerful server [15] completing the computation for the client.

In this thesis we are concerned with an encryption framework called *Generalised Encryption Scheme* (*GES*) [16]. This model describes the *class of encryption protocols* where the classical client derives the result of a difficult computation using the server's ability to efficiently solve computationally infeasible problems, while at the same time protecting the privacy of his input. Therefore, all the deduced results in this paper do not refer to a specific encryption protocol, but to an entire family of encryption schemes which solve the secure delegated computation task.

Our target is to study what types of problems could be solved by a client in the *Generalised Encryption Scheme* scenario. In other words, we examine the relation between solving a problem and the complexity of encrypting it using this framework. Thus, besides the practical motivation of securely using the resources of a quantum computer to obtain the outcome of a hard computation, the *Generalised Encryption Scheme* also leads to important results in the field of *Complexity Theory* [17].

In ***Section 4*** we will review a series of essential theorems proved by Martin Abadi [16] regarding the implications of encrypting a problem, which emphasize the idea that the harder a problem is, the more difficult it is to encrypt without revealing anything about the input. The fundamental result obtained by Abadi shows that there is a strong connection between *encryption* and *nonuniform complexity* [17, 18]. More precisely, we have that any *encryptable* function can be evaluated by a *nondeterministic* algorithm receiving an external *polynomial advice*.

The **main contribution** of our project is the investigation, based on the aforementioned result, whether problems which are efficiently computed by a quantum machine [19] (BQP problems) admit a *Generalised Encryption Scheme*.

But why are we interested if we can resolve problems from this particular complexity class?

First of all, BQP is the most important and well-known quantum complexity class. It refers to the decision problems which can be solved with high probability by a quantum computer in polynomial time [1, 19]. Because quantum computing is inherently probabilistic, we can view the class BQP as the quantum analogue of the tractable classical classes P or BPP [17]. Secondly, there are not many known precise relations between quantum and classical complexity. Abadi's results [16] tell us that we can define an encryption scheme for any problem solvable in polynomial time and at the same time that there can be no such scheme for the "hardest" problems solvable by a nondeterministic algorithm. However, because there is no acknowledged complexity connection between the classes NP and BQP [20], we do not know whether there exists an encryption scheme for every BQP problem.

What we want to show is the following *no-go result*: ***no*** $\boldsymbol{BQP - Hard}$ ***problem can be encrypted using the Generalised Encryption Scheme***. Considering Abadi's central theorem, proving this statement is equivalent to showing that the class BQP cannot be included in the class $NP/poly$ ($NP/poly$ [18, 20] refers to problems solved by a nondeterministic algorithm using an external polynomial advice).

Of course, proving such a statement is at least as difficult as proving that P is not contained in NP [17, 21], so instead, we give *strong evidence* for this extremely important complexity theory result by **relativizing** the relation between the 2 classes [22]. This means that we indicate the separation between the 2 complexity classes with respect to an oracle (the machines from the 2 classes are allowed to make queries to an oracle - which is a fixed language).

In ***Section 7*** we describe a candidate problem we found for the oracle separation, named *Simon's* Problem [23]. We start by proving that the problem can be solved by a BQP algorithm.

Then, using a diagonalization approach [20] we show that we can build an oracle O such that *Simon's* Problem cannot be solved by a nondeterministic polynomial time algorithm which has access to O.

The second part consists of indicating that the no-go result holds even when the algorithm receives an extra polynomial size advice [24].

For this proof we use an *innovative technique*. We build a matrix where each line refers to a possible advice received by a nondeterministic machine, each column refers to a possible input of the problem and the cell of the matrix on position (i, j) specifies if the machine receiving advice i accepts or not the input j. An advice is considered good if the machine receiving that advice decides correctly any input. Using this method, we intend to show that we can construct an oracle, such that for every input, a fraction of the total number of advices are not good, so we can eliminate them. Consequently, when we reach the last input we can conclude that no possible polynomial advice can help a nondeterministic machine to always decide correctly, which ends our proof.

In ***Section 8*** we study another candidate problem for the separation between the quantum and the classical complexity class, called the *Forrelation* Problem [25]. This is an even more interesting problem as it belongs to the class $BQP - Complete$. Using a series of polynomial reductions between different problems, we end at a problem which is easier to solve than *Forrelation* called *Distribution*

Correlation. For *Distribution Correlation* [25] we can follow the same proof we used for *Simon*'s problem, which implies that we can draw the same **no-go conclusion** for the *Forrelation* problem.

Our **second main contribution** is related to a quantum protocol named *Universal Blind Quantum Computing* (*UBQC*) [26] described in ***Section 5***. This protocol has the same aim as the *Generalised Encryption Scheme*, which is, allowing a client to solve a hard problem by communicating with a quantum server. The main differences from *GES* are that the client is no longer entirely classical, but must have minimal quantum abilities and the problem itself must be hidden from the server [26, 27].

Starting from this protocol we investigate the existence of a classical version of *UBQC* (*CUBQC*), namely a protocol where the computation is also encrypted, but where the client is entirely classical and the interaction between the client and the quantum server is classical too. Moreover, *CUBQC* must satisfy the correctness and security conditions imposed by the quantum protocol *UBQC.*

The motivation behind this new protocol can be illustrated in the following example. Suppose we have a standard computer and would like to use a quantum server to compute for us the longest path in a graph G. Then, this encryption protocol would guarantee us that we would get the correct answer without the quantum computer inferring the input G, or even the fact that he was actually performing a "graph longest path" task.

The **novelty** regarding this topic consists of proving a connection between *CUBQC* and *GES*.

Specifically, in ***Section 6*** we show that we can represent *CUBQC* as an instance of the *GES* framework, which meets the conditions set by *UBQC*. This proof implies that any result regarding what class of problems can or cannot be solved in the *GES* scenario also applies to the *CUBQC* protocol. Thus, if *GES* cannot solve *BQP-Hard* problems we have the same no-go result for *CUBQC*.

Because our major proofs and results are correlated with the field of *Complexity Theory*, we will give in ***Section 3*** an extensive description of the necessary *Complexity Theory* background. We will put accent on the two models of computation which are most relevant to our project: *Oracle Turing Machines* [20, 22] and *Advice Turing Machines* [24, 28]. We are going to outline examples of problems and well-known theorems regarding these two classes and also illustrate some proof techniques useful to our contributions.

Section 2 presents an overview of the principles which lie at the basis of Quantum Computation.

The subchapters presenting the *quantum operators*, *gates* and *measurements* [29] are necessary for the understanding of the proof that a particular problem belongs to the quantum class BQP. On the other hand, the *MQBC* subsection [6, 29] is not required for perceiving our results, but is useful in understanding how the *UBQC* quantum protocol works.

2. Quantum Computing

The central unit of quantum information and computation is the *qubit* [2, 3, 5], the quantum analogue of a classical bit. To physically implement qubits many different systems can be used including atoms, ions, nuclear spins and photons [1]. The classical bits can take 2 possible values, 0 or 1. Qubits analogous expression are the states $|0\rangle$ and $|1\rangle$. However, the crucial difference is that a qubit can also act as if it is in the 0 and 1 state at the same time [3, 4]. Specifically, the qubit state can be described as a linear combination of $|0\rangle$ and $|1\rangle$. We can get a better picture of this statement by considering the physical representation of a qubit: for a particle, its spin state can be aligned up (corresponding to state $|1\rangle$), down (corresponding to state $|0\rangle$) or be arbitrarily aligned, in between these 2 states (a linear combination of the up and down states) [1, 4]. Therefore, any qubit $|\psi\rangle$ can be written as a *superposition* of $|0\rangle$ and $|1\rangle$: $|\psi\rangle = a|0\rangle + b|1\rangle$, where $a, b \in \mathbb{C}$ are also known as *amplitudes* [2].

In fact, $|\psi\rangle$ is a vector in a complex Hilbert Space $\mathcal{H}$ of dimension 2, while the $|0\rangle$ and $|1\rangle$ states are vectors representing the coordinate axes of this complex space. For this reason, we say that $(|0\rangle, |1\rangle)$ constitute a computational basis [1, 3] for any qubit.

Given a classical bit we can decide if it is 0 or 1 by checking its value. On the other hand, for its quantum counterpart we cannot determine the pair of amplitudes (a, b) associated with its state [2, 28]. This is because if we measure $|\psi\rangle$ then we always obtain either the value 0 or 1. Basically, a measurement would destroy the quantum state $|\psi\rangle$ and collapse it to either the state $|0\rangle$ or $|1\rangle$. Moreover, the result of the measurement is nondeterministic, we get the result 0 with probability $|a|^2$ and the result 1 with probability $|b|^2$. Since we are talking about probabilities, the 2 values should sum up to 1, and we have that $|a|^2 + |b|^2 = 1$. When we have this property for $|\psi\rangle$, we say this quantum state is a *pure* state [5]. In this way, we know that the norm of the vector state $|\psi\rangle$ is equal to 1 ($|\psi\rangle$ is a unit vector of the $\mathcal{H}$ space).

This main difference between the standard state of a classical bit and the "unusual" superposition of a quantum bit can be explained through the next experiment. Imagine the bit as an ordinary coin. The value of the bit represents the result of tossing that coin: heads → 0 or tails → 1. On the other hand, the qubit can also reveal the status of the coin while it is in the air. At this moment, the coin is neither heads nor tails and is rather in a continuum process alternating between the 2 values, described as a superposition of the heads and tails states (for instance $|\phi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$). Then, after the coin has landed, but before we actually see it, we know that with probability $\left(\frac{1}{\sqrt{2}}\right)^2$ the coin is heads and with probability $\left(\frac{1}{\sqrt{2}}\right)^2$ is tails. Finally, when we look at the coin, equivalent to making the measurement, we observe the result: either heads (state $|0\rangle$) or tails (state $|1\rangle$).

For a better understanding of the qubit concept we can also view it from a geometric perspective [3].

Let $|\psi\rangle \in \mathcal{H}$ be a quantum pure state: $|\psi\rangle = a|0\rangle + b|1\rangle$ with $|a|^2 + |b|^2 = 1$. Then, we can express $|\psi\rangle$ as:

$|\psi\rangle = \cos\frac{\theta}{2} \cdot |0\rangle + e^{i\varphi} \cdot \sin\frac{\theta}{2} \cdot |1\rangle$, where $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$.

The Hilbert Space $\mathcal{H}$ can be represented as a 2-sphere (Figure 1) and each pure state such as $|\psi\rangle$ is a unit vector situated on the boundary of this 2-sphere, whose north and south pole correspond to the computational basis states $|0\rangle$ and $|1\rangle$.

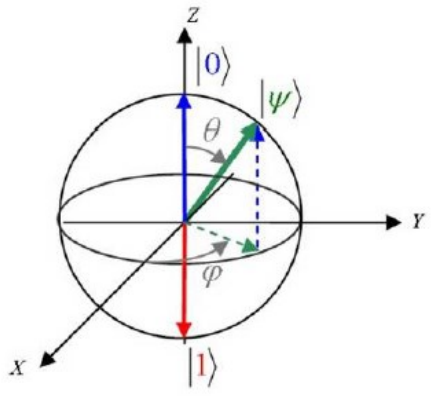


Figure 1. 2-sphere representation of Hilbert Space. Source [29]

An essential question we need to address is related to the amount of information which can be stored in a qubit [13]. We might be tempted to think that since the sphere contains an unbounded number of unit vectors, each associated with a different pair (θ_i, φ_i), we could use as storage space all the bits required for the unbounded binary representation of θ_i or φ_i. Unfortunately, as shown before, the measurement makes the state $|\psi\rangle$ collapse to $|0\rangle$ or $|1\rangle$, so we just get only one bit of information[1].

Consequently, in order to have access to more information, we need to consider systems consisting of several qubits.

Such a composite system is defined as the *tensor product* [4, 26] between individual smaller quantum systems. Essentially, the tensor product is an operation which allows 2 different vector spaces to join together and form a bigger vector space. For instance, if we have the vector space A with dimension m and the vector space B with dimension n, then the tensor product between A and B, notated as $A \otimes \mathrm{B}$, is a vector space of dimension $m \cdot n$. A quantum state from $A \otimes \mathrm{B}$ is just a *superposition* of states $|a_i\rangle \otimes |b_i\rangle$ (which is usually written $|a_i b_i\rangle$), where $|a_i\rangle$ is a vector from A and $|b_i\rangle$ is a vector from B. Therefore, if we have the orthonormal basis $\{|i_1\rangle, |i_2\rangle, \dots, |i_m\rangle\}$ for A and the orthonormal basis $\{|j_1\rangle, |j_2\rangle, \dots, |j_n\rangle\}$ for B, then the set $\{|i_1\rangle \otimes |j_1\rangle, |i_1\rangle \otimes |j_2\rangle, \dots., |i_m\rangle \otimes |j_n\rangle\}$ forms an orthonormal basis for $A \otimes \mathrm{B}$.

Take the case of the vector space composed of quantum states containing 2 qubits. This vector space can be written as $A \otimes \mathrm{B}$, where $A = B =$ 2-dim $\mathcal{H}$. As a result, any 2-qubit quantum state $|\psi\rangle$ can be written as a linear combination of the states $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$: $|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$. $\rangle$. Exactly as in the 1-qubit scenario, after the measurement $|\psi\rangle$ will collapse to the state $|ij\rangle$ with probability a_{ij}, $i, j \in \{0,1\}$.

An important state containing 2 qubits is $|\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, called the Bell state [3, 5], which introduces the concept of *entanglement* [2]. This notion refers to a quantum state consisting of multiple qubits, which cannot be expressed as a tensor product between individual systems, but only as an entire ensemble. In fact, we can prove that the Bell's state is entangled, by showing there does not exist any single-qubit states $|x\rangle = a|0\rangle + b|1\rangle$ and $|y\rangle = c|0\rangle + d|1\rangle$, such that $|\Phi\rangle = |x\rangle \otimes |y\rangle$.

The fundamental property of entangled qubits is that measurements performed on them appear to be correlated as indicated in the following scenario.

Suppose we draw apart the 2 qubits of the entangled quantum state $|\Phi\rangle$ in 2 locations extremely far from each other. If we measure the first qubit of $|\Phi\rangle$, with probability $\frac{1}{2}$ we get the outcome 0 and the state after the measurement becomes $|\Phi\rangle = |00\rangle$ and with probability $\frac{1}{2}$ we obtain the result 1 and the state after the measurement becomes $|\Phi\rangle = |11\rangle$. After this, if we measure the second qubit we would normally expect the result of this measurement to also be completely random. However, it turns out that the outcome of the measurement on the second qubit is equal to the measurement result we got for the first qubit. The correlation is also visible if we try different kinds of measurements [4, 8] on the 2 qubits: there are still connections between the outputs of the measurements on the first and second qubit.

2.1. Quantum Operators

Starting from a composed quantum state, we need to determine a way to manipulate its constituent qubits in order to perform certain computations [6]. This thing is achieved using *operators*.

Essentially, an *operator* A is a function from a vector space V of dimension n to a vector space W of dimension m [29]. We can view A as a matrix of m lines and n columns. Then, given an orthogonal basis $(|v_1\rangle, |v_2\rangle, \dots, |v_n\rangle)$ for V and an orthogonal basis $(|w_1\rangle, |w_2\rangle, \dots, |w_n\rangle)$ for W, the operator A acts in the following way:

$A|v_i\rangle = \sum_{j=1}^{m} A_{j,i}|w_j\rangle \qquad \forall\, i \in \{1, \dots, n\}$

One important operator from any vector space V to itself is the identity vector I, which applied on any quantum state $|v\rangle$ behaves like this: $I \cdot |v\rangle = |v\rangle$.

Another well-known operator is the *outer product* [1]. Consider 2 quantum states $|v\rangle \in V$ and $|w\rangle \in W$. Then, the outer product between $|v\rangle$ and $|w\rangle$, notated as $|w\rangle\langle v|$ (where $\langle v|$ is the dual[1] vector of $|v\rangle$) is an operator mapping vectors from V to vectors from W:

$(|w\rangle\langle v|)|z\rangle = |w\rangle \cdot \langle v|z\rangle = \langle v|z\rangle \cdot |w\rangle \in W\ \forall\ |z\rangle \in V$, as $\langle v|z\rangle$ is just a complex number.

For any quantum operator A we define an operator named the *adjoint*, A^+, which satisfies the property that for any 2 vectors $|a\rangle$ and $|b\rangle$ from a vector space V we have that:

- the inner product between $|v\rangle$ and $A|w\rangle$, is equal to the inner product between $A^+|v\rangle$ and $|w\rangle$, $(|v\rangle, A|w\rangle) = (A^+|v\rangle, |w\rangle)$.

The operator A is *self-adjoint* if it satisfies the condition $A = A^+$.

An essential subset of self-adjoint operators refers to *projectors* [1, 4], which are mostly used in the context of state measurements. Such an operator P, projects a vector $v \in V$ to a vector $w \in W$, where W is a subspace of V. Additionally, P satisfies the property that: $P^2 = P$. Then, given the computational basis $(|1\rangle, |2\rangle, \dots, |m\rangle)$ for the vector space W, the projector $P : V \to W$ is defined as:

$P = \sum_{i=1}^{m} |i\rangle\langle i|$.

[1] The inner product of 2 vectors $|v\rangle$ and $|w\rangle$ returns a complex number written as $(|v\rangle, |w\rangle)$ or as $\langle v|w\rangle$. The dual of a vector $|v\rangle \in V$ is a function from V to $\mathbb{C}$ defined as: $\langle v| = (|v\rangle, \cdot)$ which generates the inner product between $|v\rangle$ and any vector from V.

For example, when V is the 2-dimensional Hilbert space and $P = |1\rangle\langle 1|$, for every $|\psi\rangle \epsilon\, V$, $|\psi\rangle = a|0\rangle + b|1\rangle$ we have:

$P|\psi\rangle$ = $(|1\rangle\langle 1|)(a|0\rangle + b|1\rangle) = b|1\rangle$. Therefore, P is projecting any vector from V to the subspace generated by the vector state $|1\rangle$.

A *unitary* operator is a function $U : V \rightarrow V$ such that $U^{+}U = I$. One important property of this type of operator is that the application of a unitary on two vectors does not affect their inner product [3]:

Consider 2 vectors from V, $|v_1\rangle$ and $|v_2\rangle$. We compute the inner product between $U|v_1\rangle$ and $U|v_2\rangle$:

$$(U|v_1\rangle, U|v_2\rangle) = (U|v_1\rangle)^{+}(U|v_2\rangle) = \langle v_1|U^{+}U|v_2\rangle = \langle v_1|I|v_2\rangle = \langle v_1||v_2\rangle = (|v_1\rangle, |v_2\rangle)$$

This characteristic of unitary operators tells us that if we have an orthonormal basis for V, $B = (|v_1\rangle, |v_2\rangle, \ldots, |v_k\rangle)$, then the set $B' = (|v_1{}'\rangle, |v_2{}'\rangle, \ldots, |v_k{}'\rangle)$ obtained as $|v_i'\rangle = U|v_i\rangle\ \forall i \,\epsilon\, \{1, \ldots, k\}$, is also an orthonormal basis for V, because the inner product between any 2 vectors $|v_i{}'\rangle$, $|v_j{}'\rangle$ of B' is equal to the inner product of the correspondent vectors from B, $|v_i\rangle$ and $|v_j\rangle$.

2.2. Quantum circuit model

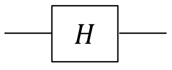
The *quantum circuit model* [6] defines a mechanism to implement any possible quantum computation.

In the classical paradigm we have the boolean circuit model which allows us to determine the value of any boolean function f by using a set of gates. These gates acting on a small number of bits, are applied sequentially in such a way that the resulting circuit can simulate the behaviour of f.

Similarly, in the quantum case, we have the analogous *quantum gates* used to construct *quantum circuits* [1, 5]. Every quantum gate can be described by a unitary operator U. This fact reveals one major difference between classical and quantum gates: quantum gates are *reversible*, while their classical counterparts are not. This property refers to the ability to rebuild the input after the application of an operator on it. It is easy to observe that from the result of an OR-gate applied on 2 bits, we cannot recover the values of the input bits. On the other hand, using the fact that every unitary operator is invertible: $U^{-1} = U^{+}$, we are able to reconstruct any input by just applying the inverse of that operator.

The most common quantum gates receive as input 1 or 2 qubits. Because a quantum state is a vector in a complex Hilbert Space, we can represent any quantum gate applied on it as a *matrix*. For instance, a quantum gate acting on one qubit is a $2x2$ matrix and in general, a quantum gate acting on n qubits is written as a 2^nx2^n matrix. This also brings an important remark, namely that the number of qubits given as input to a quantum gate must be equal to the number of qubits representing the output [29].

We will now enumerate some of the most common quantum gates which we are going to use in this project [1, 8]:

- The *Hadamard gate H* operates on single qubits and maps the $|0\rangle$ state to the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and the $|1\rangle$ state to $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Its matrix form is $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, while in a quantum circuit it is represented as: 

- The X *gate* is the analogue of the classical NOT gate, mapping $|0\rangle$ to $|1\rangle$ and $|1\rangle$ to $|0\rangle$.

It is represented as the matrix $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

- The Y *gate* applies to a qubit a π-rotation around the *Y-axis* in the $\mathcal{H}$ space, mapping $|0\rangle$ to $i|1\rangle$ and $|1\rangle$ to $-i|0\rangle$. It is represented as the matrix $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
- The Z *gate* applies to a qubit a π-rotation around the *Z-axis* in the $\mathcal{H}$ space, mapping $|0\rangle$ to $|0\rangle$ and $|1\rangle$ to $-|1\rangle$. It is represented as the matrix $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- The *Phase Shift gate*, R_θ, affects the phase of a qubit state, mapping $|0\rangle$ to $|0\rangle$ and $|1\rangle$ to $e^{i\theta}|1\rangle$. It is represented as the matrix $R_\theta = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$
- The *Controlled-NOT gate*, $CNOT$ is applied on 2 qubits and has the following result: $CNOT\ (|x\rangle \otimes |y\rangle)\ =\ |x\rangle \otimes |(x\ +\ y)\ mod\ 2\rangle$. Its matrix representation is:
  $CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
- The $SWAP$ *gate* switches 2 qubit states: $SWAP(|x\rangle \otimes |y\rangle)\ =\ |y\rangle \otimes |x\rangle$. Its matrix form is:
  $SWAP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ while its quantum circuit representation is:

- The $CSIGN$ *gate* (or *Controlled-Z*) acts on 2 qubits in the following way:
  $CSIGN(|x\rangle \otimes |y\rangle)\ =\ (-1)^{x \cdot y}\ (|x\rangle \otimes |y\rangle)$. It is represented as: $CSIGN = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$;
- The $CCSIGN$ (*controlled-controlled sign*) operates on 3 qubits:
  $$CCSIGN\ [|x\rangle \otimes |y\rangle \otimes |z\rangle]\ =\ (-1)^{x \cdot y \cdot z}\ [|x\rangle \otimes |y\rangle \otimes |z\rangle]]$$

Now, what we would like to achieve using the quantum circuit model, is to be able to compute any possible function. This property of a computational model, called *universality*, is accomplished using *universal sets of gates* [6]. A *universal set of gates* guarantees us that using a subset of quantum gates chosen from this set we are able to approximate any unitary operator.

Because the total number of quantum operators is infinite we cannot hope to precisely replicate the behaviour of any operator using only a finite set of quantum gates, but instead, we can reach a good approximation of them [7, 10]. Such an example of finite universal sets is: $\left\{CNOT, H, R_{\frac{\pi}{4}}\right\}$

Alternatively, if we would use an infinite universal set of gates, then we could obtain the exact output retrieved by any unitary operator. For example, the set containing the quantum gate $CNOT$ along with any unitary operator acting on 1 qubit, or the set consisting of the gates X, $CNOT$ and R_θ for any possible angle θ, are both *infinite universal sets*, able to simulate any possible computation.

After we have designed the quantum circuit which implements a specific computation, we take an input consisting of a number of qubits and sequentially apply the gates according to the scheme of the circuit [3, 29]. After this stage, we need to get the actual classical outcome of the computation by *measuring*(observing) the final qubits.

We will next give a more detailed description of the *quantum measurement process*.

2.3. Quantum Measurements

So far, we have only discussed about measuring a qubit in the computational basis $\{|0\rangle, |1\rangle\}$ i.e. giving outcome either $|0\rangle$ or $|1\rangle$. In fact, a measurement can be made in any *basis* $B = \{|b_0\rangle, |b_1\rangle, \ldots, |b_{k-1}\rangle\}$, which is a set of k linearly independent vectors and k is the dimension of the Hilbert space of our states [6, 13]. Any state $|\psi\rangle$ can be written as a linear combination of vectors from B, $|\psi\rangle = c_0|b_0\rangle + c_1|b_1\rangle + \cdots + c_{k-1}|b_{k-1}\rangle$. As a result, a measurement made in the basis B would yield as result one of these vectors $|b_i\rangle$ with the corresponding probability c_i^2.

For example, take the quantum state $|\psi\rangle = a|0\rangle + b|1\rangle$. Then, by measuring in the computational basis we obtain that $|\psi\rangle$ is in state $|0\rangle$ with probability $|a|^2$ and in state $|1\rangle$ with probability $|b|^2$. But, suppose we choose to measure $|\psi\rangle$ in the basis $B' = \{|+\rangle, |-\rangle\}$ where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. We can rewrite $|\psi\rangle$ as $|\psi\rangle = \frac{a+b}{\sqrt{2}}|+\rangle + \frac{a-b}{\sqrt{2}}|-\rangle$. Therefore, after a measurement in basis B', we obtain that that $|\psi\rangle$ is in state $|+\rangle$ with probability $\frac{|a+b|^2}{2}$ and in state $|-\rangle$ with probability $\frac{|a-b|^2}{2}$.

Now, consider a general quantum state $|\phi\rangle = \sum_{i=0}^{m-1} a_i|i\rangle$. By measuring this state we get one of the m possible outputs $\{0, 1, \ldots, m-1\}$, each having a probability $P(i)$.

For every possible measurement result we can associate an operator M_i, where M_i is defined on the vector space of $|\phi\rangle$. Then, the probability of outcome i is:

$$P(i) = \langle\phi|M_i^{+}M_i|\phi\rangle$$

The *measurement operators* [1] must satisfy the property that: $\sum_{i=0}^{m-1} M_i^{+}M_i = 1$. In this way, we are assured that the sum of probabilities of all results is equal to 1:

$$\sum_{i=0}^{m-1} P(i) = \sum_{i=0}^{m-1}\langle\phi|M_i^{+}M_i|\phi\rangle = \langle\phi|\left(\sum_{i=0}^{m-1} M_i^{+}M_i\right)|\phi\rangle = \langle\phi||\phi\rangle = 1$$

After the measurement is performed, the original quantum state $|\phi\rangle$ is also modified depending on the outcome of the measurement [2, 29]. Therefore, supposing the outcome was j, the state $|\phi\rangle$ will now be equal to $|\phi'\rangle = \frac{1}{\sqrt{P(j)}}M_j|\phi\rangle$.

Example. Take $|\phi\rangle$, a unit norm vector from the 2-dim $\mathcal{H}$ defined as: $|\phi\rangle = a_0|0\rangle + a_1|1\rangle$, such that $|a_0|^2 + |a_1|^2 = 1$.

Then, for the outcome measurement 0 we define the operator $M_0 = |0\rangle\langle 0| = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$ and similarly, for outcome measurement 1 we define the operator $M_1 = |1\rangle\langle 1| = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$.

We observe that both M_0 and M_1 are self-adjoint projectors: $M_0 = M_0^{+} = M_0^2$ and $M_1 = M_1^{+} = M_1^2$

Consequently, we can compute the probabilities for the 2 possible outcomes:

$$P(0) = \langle\phi|M_0^{+}M_0|\phi\rangle = (\langle 0|\overline{a_0} + \langle 1|\overline{a_1})M_0(a_0|0\rangle + a_1|1\rangle) = \overline{a_0}\cdot a_0 = |a_0|^2$$

$$P(1) = \langle\phi|M_1^{+}M_1|\phi\rangle = (\langle 0|\overline{a_0} + \langle 1|\overline{a_1})M_1(a_0|0\rangle + a_1|1\rangle) = \overline{a_1}\cdot a_1 = |a_1|^2$$ [2]

[22] $\bar{x}$ is the complex conjugate of x; $|x|$ is the modulus of x

And the state after the measurement becomes:

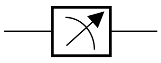
$|\phi\rangle \rightarrow\ |\phi'\rangle = \frac{1}{\sqrt{P(0)}} M_0|\phi\rangle = \frac{1}{|a_0|} \cdot (a_0|0\rangle)$, if the measurement result is 0, or

$|\phi\rangle \rightarrow\ |\phi'\rangle = \frac{1}{\sqrt{P(1)}} M_1|\phi\rangle = \frac{1}{|a_1|} \cdot (a_1|1\rangle)$, if the measurement result is 1.

This was an example of a *projective measurement* [10] where all operators M_i are projectors. For this type of measurement, we observe thatM_i satisfy the probability condition specified above:

$\sum_{i-0}^{m-1} M_i^+ M_i = \sum_{i-0}^{m-1} M_i = 1.$

As we have seen, the new state after the measurement depends on the measurement result. Hence, in the time between the measurement is performed and the time we actually observe its result, the quantum state is known to be in a superposition of m states, each associated with a probability $P(i)$. At that point, we can conclude that $|\phi\rangle$ is in the $\frac{1}{\sqrt{P(j)}} M_j|\phi\rangle$ state, with probability $P(j)$.

In a quantum circuit, the symbol used for the measurement of a qubit is: 

2.4. Measurement Based Quantum Computing

Measurement Based Quantum Computing(MBQC) is a model for quantum computing, equivalent to the circuit model [1, 4, 29]. While in the quantum circuit paradigm we applied a sequence of operators and then performed a measurement to obtain the classical result of the computation, in *MBQC* we execute a sequence of measurements to reach the desired result.

The starting point of a computation in the *MBQC* model is a *graph state*, which is an entangled state containing a large number of qubits [6]. This graph state must be *generic* to allow for a wide range of computations to be completed from this pattern. Then, we perform measurements at proper basis on the qubits of this state in order to derive a certain computation. In this way, the initial entanglement of the graph state is destroyed by the measurements.

Given a graph G with V its set of vertices and E its set of edges, we build the graph state $|G\rangle$ in the following manner. Every vertex of G is associated with a qubit in the $|+\rangle$ state and every edge of G is associated with a pair of entangled qubits, to which we apply the *Controlled-Z* operator. Consequently:

$|G\rangle = \prod_{(i,j)\,\epsilon\,E}(C - Z|ij\rangle) \cdot\ |+\rangle^{|V|}.$

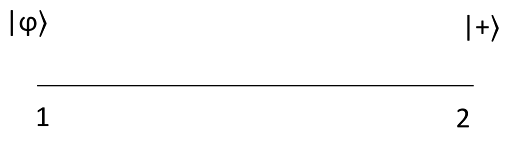
The qubits in the graph are either measured in the $\{|0\rangle, |1\rangle\}$ or in the $\{|+_\theta\rangle, |-_\theta\rangle\}$ basis (where $|+_\theta\rangle\ =\ \frac{1}{\sqrt{2}}(|0\rangle +\ e^{i\theta}|1\rangle)$ and $|-_\theta\rangle\ =\ \frac{1}{\sqrt{2}}(|0\rangle - e^{i\theta}|1\rangle)$).

Measurements in the base $\{|+_\theta\rangle, |-_\theta\rangle\}$ are used to implement computations. On the other hand, we might need to *reduce* the graph state. For instance, if we want to compute an operator acting on a single qubit, we would require the graph to be in a 1-dimensional space. Removing qubits from the initial graph state in order to match the appropriate design pattern for our computation, is achieved through measurements in the $\{|0\rangle, |1\rangle\}$ basis [29].

Now, we will focus on how does this model allow us to complete certain computations using $\{|+_\theta\rangle, |-_\theta\rangle\}$ measurements.

For every single-qubit measurement, we decide upon a measurement angle θ based on the result of the previous measurements. Even though the outcomes of measurements are probabilistic we can modify the angles to make the final result of the computation deterministic [1, 6].

For example, if we want to compute the operator $J(\theta) = HR_\theta$, we can define the following *MBQC* pattern:



The graph consists of 2 nodes, where the first one represents the input for the computation, $|\phi\rangle = a|0\rangle + b|1\rangle$.

Therefore, the graph state becomes:

$$|G\rangle = C - Z(|\phi\rangle|+\rangle) = C - Z\left((a|0\rangle + b|1\rangle)\,|+\rangle\right) = a|0\rangle|+\rangle + b|1\rangle|-\rangle$$

Now, we measure the first qubit in the $\{|+_\theta\rangle, |-_\theta\rangle\}$ basis. The measurement projectors are:

$P_{+\theta} = |+_\theta\rangle\langle+_\theta| = \frac{1}{2}(|0\rangle\langle 0| + e^{-i\theta}|0\rangle\langle 1| + e^{i\theta}|1\rangle\langle 0| + |1\rangle\langle 1|)$ and

$P_{-\theta} = |-_\theta\rangle\langle-_\theta| = \frac{1}{2}(|0\rangle\langle 0| - e^{-i\theta}|0\rangle\langle 1| - e^{i\theta}|1\rangle\langle 0| + |1\rangle\langle 1|)$

Then, if the outcome r of the measurement equals to 1, our state becomes:

$P_{+\theta}|G\rangle = |+_\theta\rangle(a|+\rangle + be^{-i\theta}|-\rangle)$

But, we observe that the result of applying $J(-\theta)$ on the input state $|\phi\rangle$ is: $J(-\theta)|\phi\rangle = (a|+\rangle + be^{-i\theta}|-\rangle)$, which is exactly what we have obtained at the second qubit of our state.

Therefore, if the outcome of the measurement is 1, the result of this *MBQC* pattern is $J(-\theta)$.

Otherwise, if the result r of the measurement is 0, our state becomes:

$P_{-\theta}|G\rangle = |-_\theta\rangle(a|+\rangle - be^{-i\theta}|-\rangle)$

Using the relation $XJ(-\theta)|\phi\rangle = a|+\rangle - be^{-i\theta}|-\rangle$, we observe that in this case the outcome of the *MBQC* pattern is the application of the $XJ(-\theta)$ gate on the second qubit.

Thus, to obtain the desired result we should apply the operator X^r. If the measurement result is 0, we get $X^0 J(-\theta) = J(-\theta)$ and if the result is 1, we obtain $X \cdot XJ(-\theta) = J(-\theta)$.

So, this means that using this *MBQC* pattern we can simulate the operator $J(\theta)$ for any angle θ. Given the fact that every operator U acting on a single qubit can be written as $U = J(0) \cdot J(\theta_1) \cdot J(\theta_2) \cdot J(\theta_3)$, we obtain that the *MBQC* model allows us to simulate any single-qubit operator U.

Moreover, we can also simulate the *Controlled-Z* gate by just setting an edge between 2 input states. Now, because the set of gates {*Controlled-Z*, $J(\theta)$} is universal, this tells us that the in *MBQC* we can implement any possible computation [30].

One thing we observed in the example with the operator $J(\theta)$ is that depending on the result of a measurement, we might need to apply a correction operator (an X gate in our example). Because the measurement results are not known at the moment when we define the *MBQC* scheme for a computation, we will instead modify the angles for the future measurements (for instance, measure the next qubit in the rotated basis if $r = 1$).

Adapting the angles of the measurements in order to overcome the randomness of the measurements outcomes, is based on the following relations [3]. Any measurement M_i in the $\{|+_\theta\rangle, |-_\theta\rangle\}$ basis adds the operator X^r (r the result of M_i) for all the following qubits and the Z^r operator for all the qubits situated in graph at distance = 2 from qubit i. Also, any measurement M_j in the $\{|0\rangle, |1\rangle\}$ basis adds the operator Z^r (r the result of M_j) on all the qubits which are neighbours to j.

Consequently, we set for every qubit i in the graph an initial angle α_i which allows us to perform a specific computation. An X operator applied on qubit i will change its initial angle α_i to -α_i, whereas a Z operator will change α_i to α_i + π. Therefore, for any qubit i in the graph we cumulate all the angle modifications resulting from the previous measurements and we obtain that the new angle is: $\alpha_i' = (-1)^s \alpha_i + t\pi$, where s = number of qubits where an X correction is applied, t = number of qubits where the Z correction is applied [4, 29].

3. Complexity Theory

After we presented an introduction to the field of Quantum Computing, we cover some necessary background in *Complexity Theory*, which is essential for the understanding of our project.

We proceed by defining the complexity classes, both classical and quantum, which are going to be encountered in our proofs [17, 20, 21].

- P is the class of problems which can be solved *efficiently in polynomial time* by a *deterministic Turing Machine* [17]. We say that a language L belongs to the class P if we can build a deterministic Turing Machine M running in polynomial time, such that for every $x \epsilon L$, M accepts x (returns 1) and for every $x \notin L$, M rejects x (returns 0).
- NP is the class of problems which can be computed in *polynomial time* by a *nondeterministic Turing Machine* [20]. Every NP algorithm has 2 stages: the first in which he makes a *guess* for the result of the problem and the second where he *verifies* in polynomial time if the guess was a correct answer. A useful way to describe the behaviour of any NP algorithm is by looking at its *computational tree* [21]. The computational tree indicates how does the algorithm work, given a specific input x to the problem. The initial configuration of the Turing Machine on the input x represents the root of the tree. Every internal node corresponds to a computation performed by the algorithm and the children of the node are the possible configurations which can be obtained in one computational step. For each nonterminal node, the choice for the next computation is made in a nondeterministic way [17].

In a *decision problem*, the leaves of the computational tree are either *accepting* or *rejecting states*. Then, we say that a string s is accepted by the language defined in the decision problem, if in the computational tree generated for s there exists at least one *path* ending in an accepting state. On the other hand, a string s is rejected if all possible paths of the tree end in a reject state [33].

- $NP-Hard$ problems are the problems *at least as difficult to solve as the hardest problems in the NP class*. We say a problem G belongs to $NP-Hard$ if every problem from NP can be reduced efficiently in polynomial time to G.
- The $NP-Complete$ class contains the problems which are both in NP and in $NP-Hard$.
- A decision problem G belongs to the class $co\text{-}NP$, if the complement of G, obtained by switching the accepting answers with the rejecting answers and vice versa, is in the NP class [20, 32]. In this way, we have that a string x is accepted if every path of the computational tree ends in an accepting state and is rejected if there exists one path ending in a rejecting state.

- The BPP class (*Bounded-error probabilistic polynomial time*) contains the problems which can be solved by *probabilistic Turing Machines* [16, 34] in polynomial time, with the probability of giving a wrong answer being less than $\frac{1}{3}$. We say a language L is in the BPP class if we can construct a probabilistic Turing Machine M running in polynomial time such that:
  - If $x \epsilon L$, then M accepts x with probability $p \geq \frac{2}{3}$
  - If $x \notin L$, then M accepts x with probability $p \leq \frac{1}{3}$
- The ZPP class (*Zero-error probabilistic polynomial time*) consists of the problems which can be solved in polynomial time by probabilistic Turing Machines which may also return the answer "*Do not know*" for some inputs. Specifically, a probabilistic Turing Machine M which solves a problem G from ZPP acts in the following way:
  - M gives the correct answer for an input x with probability $p > \frac{1}{2}$
  - M answers "Do not know" for an input x with probability $p < \frac{1}{2}$.
- The BQP class (*Bounded-error quantum polynomial time*) is the *quantum equivalent* of the BPP class. BQP consists of the problems which can be solved in polynomial time by *Quantum Turing Machines* with the probability of giving a wrong answer being less than $\frac{1}{3}$ [19]. We can prove that a decision problem G belongs to BQP by showing that there exists a *quantum circuit of polynomial size* (as defined in the quantum circuit model section) which can solve G with bounded-error [19, 30].

Now, we need to focus on 2 particular categories of problems: problems which can be solved by Turing Machines which receive an *additional "help"* called *advice* [28] and problems which can be solved by Turing Machines which have *access to an oracle* [22] that can correctly answer to particular questions. We are going to give a thorough description of these two models of computation as they are directly connected with our major results. In order to prove our main statements, we need a full understanding of the behaviour of the machines which can solve these classes of problems. Hence, together with the formal definitions, we will also present examples and results regarding the two models, which we are going to use in our proofs.

3.1. Advice Turing Machines

An *advice* function is any function $f: \mathbb{N} \rightarrow \Sigma^*$.

Turing Machines with an *advice* function f receive an additional help to solve problems, in the form of an *advice string* $f(n)$. The most important property of the *advice* function is that this external information does not depend on the value of the input for a problem, but only on the *input size* [20].

That is, for a given input x, the received advice is $f(|x|)$.

DEFINITION 1**:** The class $\mathcal{C}/\mathcal{F}$ of *languages recognized by Turing Machines with advice* is the set of languages A associated with a language C from $\mathcal{C}$ and an advice function f from $\mathcal{F}$ such that:

$A = \{x \in \Sigma^* | \; the\ pair\ (x, f(|x|)) \in C,\ C \in \mathcal{C}, f \in \mathcal{F}\}$.

In other words, we say that there exists a function f from $\mathcal{F}$, which gives the necessary additional information to a machine from the class $\mathcal{C}$, in order to accept a more difficult language belonging to the class $\mathcal{C}/\mathcal{F}$.

We will now present a series of classes of advice languages and their interpretation [17, 18, 28]:

- EXAMPLE 1: $\mathcal{F} = \ log$ represents the set of functions f satisfying the property:
  $\forall n \in \mathbb{N}\ \exists c \in \mathbb{N}$ such that $|f(n)| \leq \ c \cdot \log n$.

  We define P/log as the class of languages $L \ = \ \{x \mid (x, f(|x|)) \in A, A \in P, f \in log\}$.

  Thus, for any such language $L \in P/poly$ we can determine if a string x belongs to L by using a deterministic polynomial time algorithm with the help of an external information s, where $|s| \ = \ log(x)$.

- EXAMPLE 2: $\mathcal{F} \ = \ poly$ represents the set of functions f satisfying the property:
  $\forall n$, $\exists$ a polynomial p such that $|f(n)| \leq \ p(n)$.
  We define $P/poly$ as the class of languages $L \ = \ \{x \mid (x, f(|x|)) \in A, A \in P, f \in poly\}$.

  EXAMPLE 3: And similarly, $NP/poly$ is represented by the set of languages:

  $L \ = \ \{x \mid (x, f(|x|)) \in A, A \in NP, f \in poly\}$.

If the advice is the empty string then we obtain exactly the languages from the class $\mathcal{C}$, therefore, we obtain that $\mathcal{C} \subseteq \mathcal{C}/\mathcal{F}$, and particularly $P \ \subseteq \ P/poly$ or $NP \ \subseteq \ NP/log$.

3.1.1. $P/poly$

The external information can be an extremely powerful tool allowing a machine to decide complex languages which initially could not have been decided by the machine.

This is best exemplified by the relation between bounded error probabilistic polynomial time machines (BPP) and deterministic polynomial time machines (P). As we know, every TM in P can be simulated by a probabilistic TM in BPP, so $P \subseteq BPP$ [24].

But, we can show that by receiving a polynomial advice every deterministic machine is also able to simulate a probabilistic TM from BPP [28]:

THEOREM 1: $BPP \subseteq P/poly$.

Proof. Consider a language $L \in BPP$, decided by a BPP machine M. M receives 2 inputs: a string x and a random string r. Supposing the input x is of length n, then as M is a polynomial time machine, it runs on x in $p(n)$ time, with p a polynomial. Moreover, the length of the random string r must also less or equal than $p(n)$.

We intend to prove that we can use the random string as an advice to a deterministic polynomial time machine in order to decide L, so we must show that for every possible length m, there exists a random string r_m of size $p(m)$ for which:

$M(x, r_m) = \begin{cases} 1, & x \in L \\ 0, & x \notin \mathrm{L} \end{cases}$ $\forall x$ of length m.

We define the set of *wrong* random strings as the strings which cannot be the advice:

$wrong(x) = \{r \in \{0,1\}^{p(n)} \mid M(x,r)$ gives an incorrect answer$\}$.

Therefore, what we need to show is that if we rule out all strings from the set $wrong(x)$ for all x, we will still have at least one string left in $\Sigma^* - \bigcup_{x \in \{0,1\}^n} wrong(x)$ and this string will allow M to always give the correct answer.

Using *amplification* of BPP [21], we can obtain that the error probability of M is at most $\frac{1}{2^{2n}}$.

Then, this means that $P(r \in wrong(x),\ r \in \{0,1\}^{p(n)}) = \frac{1}{2^{2n}}$.

Generalising for all inputs x of length n and by using Boole inequality[3][34] we reach the relation:

$$\mathrm{P}\left(r \in \bigcup_{x \in \{0,1\}^n} wrong(x)\right) \leq \sum_{x \in \{0,1\}^n} P\big(r \in wrong(x)\big) = 2^n \cdot \frac{1}{2^{2n}} = \frac{1}{2^n}$$

This demonstrates that there is an exponentially small number of random strings which give the *wrong* answer. Consequently, we can choose one random string r_c from the complement of the $wrong$ set. Then, we know that $M(x, r_c)$ must be correct for any input x of length n. Therefore, if an "*adviser*" gives us this string r_c, we can use it as advice to always decide correctly if an input belongs or not to the language L. $\square$

[3] Given a set of actions $\{A_1, A_2, \dots, A_k\}$, the probability of at least one A_i taking place is less or equal than the sum of probabilities that each action takes place:

$P\big(\bigcup_{i=1}^{k} A_i\big) \leq \sum_{i=1}^{k} P(A_i)$

Another example of *non-uniform models* (where we basically have a different algorithm for every possible size of the input) refers to *Boolean Circuits* [17, 20].

DEFINITION 1.1: A *Boolean Circuit* helps us compute a binary function by sequentially applying the logical gates AND, OR, NOT on a given input. The computation is represented as a directed graph with V – the set of vertices, E – set of edges and a tag function $t : V \to \{x_1, x_2, \dots, x_m, 0, 1\} \cup \{NOT, AND, OR\} \cup \{output\}$, defined as:

$$t(v) = \begin{cases} NOT, & in - degree(v) = 1 \\ AND \mid OR, & in - degree(v) = 2 \\ x_1 \mid x_2 \mid \dots \mid x_m \mid 0 \mid 1, & in - degree(v) = 0 \ (input\ vertices) \\ output, & in - degree(v) = 1, out - degree(v) = 0 \end{cases}$$

Essentially, the graph consists of m input nodes, 1 output node (which denotes the result of the computation) and internal nodes named *gates* which are tagged with the boolean operations AND, OR, NOT (Figure 2). We can consider that each node has a value associated with it: input nodes x_i have exactly the value x_i, while for every other node, the value is obtained by applying the gate indicated by its tag to the values corresponding to its children nodes [21].

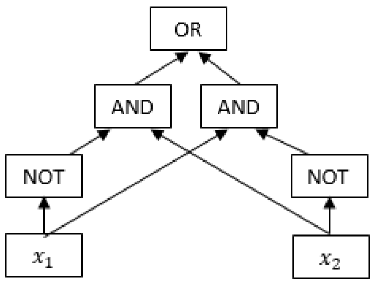


Figure 2. Example of Boolean Circuit which implements the XOR function

The connection between *Boolean Circuits* and *Advice Turing Machines* can be observed in the fact that to decide a language L, then for every possible size n of an input we need to define a new circuit which receives n inputs.

DEFINITION 1.2: We say a language L has *polynomial circuits* if there is a polynomial p and an array of circuits $\{C_n\}$ satisfying the following properties:

I. The number of gates in the circuit $C_m \leq p(m)\ \forall m$;
II. C_m receives m inputs and returns 1 if $x \in L$ and 0 if $x \notin L \ \ \forall x \in \{0,1\}^m$;

There is a *strong relation* between *polynomial circuits* and machines with *polynomial advice* [20]:
THEOREM 2: *$L \in P/poly \Leftrightarrow L$ has polynomial circuits.*

Proof. "⇒" Consider $L \in P/poly$. Then, there exists a polynomial advice function f and a set $A \in P$, such that: $x \in L \ \Leftrightarrow \ (x, \ f(|x|)) \in A$.

For every possible length n we can build a circuit C_m, which has as input nodes the bits of a input x of length n and the $|f(n)|$ bits representing the advice. This circuit just returns the pair $(x, f(|x|))$ which is given as an input to a second circuit $C_m{}'$ implementing A ($C_m{}'$ must exist, as A can be solved in polynomial time). Consequently, $C_m{}'$ will give the output 1 if $(x, f(|x|)) \in A \Leftrightarrow x \in L$ and 0 otherwise ($x \notin L$). In other words, every boolean circuit $C_m{}'$ decides L for every input x of length m. □

"⇐". Now, take a language L which has polynomial circuits. We want to build a deterministic polynomial-time machine M which decides L with the aid of a polynomial external information. We can encode each circuit C_m using the alphabet $\{0,1\}$ into a string s_m. According to the definition of L, there exists a polynomial p such that $C_m \leq p(m)$. Then we also know that the resulting encoding of C_m can be determined in polynomial time, so $|s_m| \leq p(m)$. In this way, we can use s_m as a polynomial advice for every input of size m to figure whether an input of length m is in L or not. Therefore, $L \epsilon P/poly$. □

The extra power brought by the *polynomial advice* to a *deterministic polynomial-time machine* can also be viewed in the following result [18, 24]:

THEOREM 3: *The class $P/poly$ contains undecidable languages*.

This is a remarkable fact given that a *non-recursive* (*undecidable*) language [17] is a language L for which there *does not exist* any TM which halts on any input x and decides correctly if x is or not in L.

Proof: We demonstrate our claim by first showing that every *unary language* (language in which all the strings are of the form 1^k) is in the class $P/poly$.

Take L_u, a unary language. As every element of L_u has the form 1^k, there exists a unique element of length n in L_u, $\forall n \epsilon \mathbb{N}$. Then, we can build the advice string for a deterministic polynomial time TM M in the following way: for each possible length of the input, m, we have the advice string $s_m = 1$, if $1^m \epsilon L_u$ and $s_m = 0$ otherwise.

Thus, M decides correctly for every possible input if it belongs to L_u or not.

Secondly, we show that there exist unary languages which are undecidable.

We select any undecidable language from $\{0,1\}^*$, call it L_N. Starting from L_N we construct the language:

$L_u' = \{1^{k(x)} \mid x \epsilon L_N$ and $k(x)$ is the number resulting from treating the binary string x as a number$\}$

Obviously, L_u' is a unary language. Moreover, since any machine which can decide L_u' can also decide L_N (there is a reduction from the language L_N to L_u') and since L_N is non-recursive, we deduce that L_u' is also non-recursive. □

3.1.2. P/log

In the case of *logarithmic advice*, the external information can rather be seen as an *index* of an element in a set of uniformly created elements [20].

P/log class also shows a separation from the uniform TMs, as the logarithmic advice helps the deterministic machine solve additional problems which initially could not be determined.

We will now present *an interesting proof method* [24, 30] involving the use of logarithmic advice which helped us gain intuition for our main proofs.

In this approach we will take into consideration every possible advice and then eliminate the candidates which do not lead to a correct answer. We are going to adapt this technique for our proof that a particular problem cannot be solved by any deterministic machine receiving external advice.

THEOREM 4*: If the SAT problem can be solved by a P/log algorithm, then $P = NP$.*

Proof: Consider a Boolean formula F containing n boolean variables: $x_1, x_2, \dots, x_n$.

The advice being logarithmic in the size of the input, for the formula F we get an advice s, such that: $|s| \leq c \cdot \log n$, c constant.

Because $SAT \in P/log$, there exist a language A in P such that $(F, s) \in A$.

Then, we show that we can determine if F is satisfiable using the following *deterministic polynomial time* algorithm [20].

```
for every $a \in \Sigma^*$ such that $|a| \leq c \cdot n$ do          // we try every potential logarithmic advice
    $F' = F$
    while $F'$ still contains variables:
        choose a variable $x_i$ from $F'$
        $F_1'$ = formula obtained after replacing all occurrences of $x_i$ with 0 in $F'$
        $F_2'$ = formula obtained after replacing all occurrences of $x_i$ with 1 in $F'$
        if $(F_1', a) \in A$ then $F' = F_1'$
        else if $(F_2', a) \in A$ then $F' = F_2'$
        else break;
    end while
    if $F'$ = true then halt and return "yes"                // found a satisfying assignment
end for
halt and reject                                              // no assignment has been found
```

Using the variable a we try every possible candidate for the logarithm advice for the input F. Even if at some point we find that for a string a we have $(F, a) \in A$, we don't stop here as this a may not be the correct advice (it might be the case that F is not satisfiable and as a is not the real advice for F the pair (F, a) is accepted by A). Instead, we continue assigning values to the variables in F until there are no more variables in F. Only then we will be certain that the considered a was the correct advice and that F was indeed satisfiable.

In this way, if the algorithm returns "yes", then we know for sure that F must be satisfiable. Furthermore, if F is satisfiable, then there exists a value for the variable a for which the machine A gives the correct answer.

If F contains n variables, then the presented algorithm runs in $O(n)$ (as the while loop is executed $2^{\log n}$ = n times).

Therefore, SAT can be solved by a polynomial-time algorithm. However, because $SAT \in NP$Complete [17], meaning that every NP problem can be reduced to SAT, it results that $NP = P$. □

3.2. Oracle Turing Machines

DEFINITION 2: An *oracle TM is a machine with access to an oracle which can decide if a string is in a language $O \subseteq \Sigma^*$.*

In addition to the standard TM, an oracle TM has an *oracle tape* [20] which can be used to query for an extra input and the machine will receive in only one computational step an answer specifying if that input belongs to O or not.

Every normal TM can be considered an oracle TM by setting the *empty set* $\emptyset$ as the oracle's language.

We can encode oracle Turing Machines in the same manner we do for standard TM and by conversion to the alphabet $\Sigma = \{0, 1\}^*$, we can view these encoding as natural numbers [17]. This means that the set of *oracle TM* is *countable*, so we can enumerate all these machines: $\{M_O^1, M_O^2, \ldots, M_O^n, \ldots\}$.

For any *oracle machine M_o* we are able to describe its *computational tree* as presented below.

We first consider that the oracle's language has not been set yet and we run M_o on input m. Then, we set the configuration of M_o on input m as the root of the *computational tree*. Next, every inner node of the tree represents a query made by M_o to the oracle O. For every such node we have 2 branches defining 2 different sets of computations which M_o might perform depending on the result of the query[21].So, if the queried string belongs to O, then we proceed to the computations specified in the left branch, otherwise to the computations indicated by the right branch. In the nodes where no more queries are made (the *leaves* of the tree), M_o will take a decision, if it is an *accepting state* or not[22].

Now, we set the language L for the oracle O. Consequently, we will obtain a single path π in the computation tree starting from the root and ending in an accept or reject node. For every node i the path goes to its left child if the string labelled at node i is inside the language L and goes to its right child otherwise.

Finally, we conclude that the initial input m is in the language accepted by the oracle TM M_o if this resulted path π terminates in an accepting state.

Comparing the structures of the computational trees of an oracle TM and of an nondeterministic TM (described in the introduction of the NP class), we can intuit the relation between *a nondeterministic TM* and a *deterministic oracle TM* [31]. Specifically, we can prove that we are able to simulate the behaviour of a *nondeterministic TM* on an *oracle deterministic TM* and also that the reversed relation is not true [30, 31]:

THEOREM 5: *Any nondeterministic TM can be simulated by an oracle deterministic TM.*

Proof. Let M_O be an oracle deterministic machine and M' be a nondeterministic machine. Then, we want M_O to be able to simulate M' using the oracle O. We remind that an oracle can decide any language L, so we define O in a rather unusual way:

O will contain the encodings of all computations performed by the nondeterministic algorithm of M'. In this way, by querying the oracle we can check if a specific computation was indeed performed by M', or even better if a specific configuration is valid for M'. For every nondeterministic action of M', M_O will use the oracle O to decide between the possible following steps.

This simulation procedure can be described in the next pseudocode.

```
$st$ = configuration of $M$ on input $m$ //st is the current node in the computation tree (now at the root)
$comp$ = “”                                // the computations completed so far
repeat
    if $st$ is an accepting node then
        - halt and accept
    else
        - add $st$ to $comp$
        - determine the possible continuations $C$ from $st$ given by transition relation
        - for every possible continuation $c \in C$
            $comp' = comp + c$
            query oracle $O$ to check if $comp'$ is a valid configuration of $M'$
            if the result of the query is “yes” then
                $st = c$
                break
        - if the result of all queries was “no” then
            halt and reject
```

We can observe that by querying the oracle we reproduce the nondeterminism of M'. Moreover, if M' accepts an input x, then this *deterministic algorithm* will find a path π in the computational tree of M' generated for x such that π ends in an accepting state. ☐

LEMMA 5.1: There exist deterministic oracle TMs which cannot be simulated by nondeterministic standard TMs.

Proof. The intuition behind this statement is that, in general, oracle machines can be extremely powerful computational systems [20, 21], by defining languages difficult to decide.

This can be illustrated by the fact that a *deterministic oracle* machine can decide any language L solved by a *nondeterministic* machine M' [17]. For instance, we can define the language of the oracle as the language accepted by M'. Consequently, a deterministic TM with oracle O only needs to take the input for M' and query O. Then, the answer received from the oracle is the output of M'. ☐

LEMMA 5.2: We can define an algorithm which allows the simulation of some oracle deterministic TM by standard nondeterministic machines.

Proof. Suppose we have a deterministic machine with oracle O, M_O and a nondeterministic TM M'. Then M' could copy all the deterministic computations of M_O until it makes a query to the oracle. At this point, M' can use its nondeterminism to choose the continuation associated with a response for M_O from the oracle O.

The main issue arises when the oracle queries the same input multiple times. Then, the nondeterministic machine might select different continuations (because it does not look at what choices it made previously), which leads to *inconsistency* (giving different results in response to queries for identical inputs) [22, 33].

Moreover, M' does not know the language defined by the oracle O, so the only thing he can do is consider every possible oracle which M_O might have, and accept an input x if it is accepted by any of these resulting oracle TMs. ☐

3.2.1. Examples of Oracle Turing Machines

DEFINITION 3: We define P^O as the set of problems which can be solved by a *deterministic polynomial-time* TM which *has access to the oracle* O. Similarly, we have that NP^O is the set of problems solved by a *nondeterministic polynomial-time* TM with *access to the oracle* O.

Whenever O is a language which cannot be decided by M, the oracle adds more computational power to M [31]. For instance, P^{SAT} represents the set of problems solved by a deterministic TM which also has access to an oracle capable of solving the SAT problem. This oracle will obviously help compute more functions. In general, we have:

AXIOM 6.1: $C \subseteq C^O$ for any oracle O and any complexity class C.

LEMMA 6.2: If O can be solved by a deterministic polynomial time machine we can show that: $P^O = P$.

Proof. We already know from AXIOM 6.1 that $P \subseteq P^O$ (1). Now, since $O \in P$, the verification algorithm used by the oracle to check if a string belongs to his language, is just a polynomial time procedure. Thus, we are able to convert every deterministic TM using oracle O into a different deterministic polynomial time TM. Consequently, we also have $P^O \subseteq P$ (2).

From (1) and (2) we obtain $P^O = P$. □

Adding an oracle allows us to *identify relations between different complexity classes*. Therefore, proving separations between classes of problems, with respect to a given fixed oracle represents a strong evidence that those separations would also hold in the lack of the oracle [32, 33].

THEOREM 7. We can find an oracle B such that the following crucial result holds: $P^B \neq NP^B$.

Proof. Given any language B we build the language $L_B = \{1^n \mid \exists \text{ string } s \in B \text{ such that } |s| = n\}$.

To decide if a string $s = 1^m$ is in L_B we can simply make a *choice* for a string $s \in \{0,1\}^m$ and then *verify* with the help of the oracle if $s \in B$. Therefore, we have shown that $L_B \in NP^B$, $\forall B$.

Now, we want to build the language B such that L_B cannot be decided by any deterministic TM with access to the oracle B.

The set of Turing Machines is countable, so we can list all deterministic TM [17], each of them being associated with a natural number i: $M_i \in P$. We create the oracle B in a series of steps, so that at step i we make sure that ${M_i}^B$ is not able to correctly decide L_B.

Thus, when we reach step i we already have determined if a number of strings $\{s_1, .., s_k\}$ belong to B. We pick m such that $m > |s_i| \ \forall i \in \{1, \dots, k\}$. Then, we run M_i on the input 1^m and we want M_i to return the wrong answer.

If M_i returns "yes" for 1^m, we remove all the strings of length m from the set B, so that we would have $1^m \notin L_B$.

If M_i returns "no" for 1^m, then we can add to B a string s', $|s'| = m$, such that s' has not been queried before (we can find such a string because $M_i \in P$ can query for a polynomial number of strings, but we have an exponential number of available strings of length m). Thereby, $1^m \in L_B$.

By this construction, we made sure that M_i returns the wrong answer, so we can conclude that L_B cannot be in the set P^B and hence $P^B \neq NP^B$. □

It is worth mentioning that in the case of a *nondeterministic oracle* $TM\ M'$, the computational tree of M' contains two types of nodes: nodes where the machine *queries the oracle* and *nondeterministic nodes* [20]. After setting an oracle, we define the continuation from the query nodes and we are left with the nondeterministic states. Then, M' *accepts an input* x if in this resulting computational tree there *exists one accepting path* [33].

In this project we aim to show that 2 complexity classes of problems BQP and $NP/poly \cap coNP/poly$ are different. Because there is no known relation between BQP and NP [30], we give a strong evidence for our claim by stating that they are different with respect to an oracle O [31]. Therefore, we define an oracle relative to which a problem is in the first class, but is not in the latter:

∃ Oracle O such that $BQP^O \not\subseteq NP/poly^O \cap coNP/poly^O$

Moreover, for the problems we considered to indicate the separation between the 2 classes, we use an oracle O, which instead of giving a *yes/no* answer, will return the value of a function on a given point. We can adapt the definition of an oracle to match this scenario in the following way.

The oracle will receive a string containing: an index i used to identify the desired function f_i, the length of the input for f_i, an input x and the index j of a bit. The oracle will return a $0/1$ answer representing the j-th bit of the value of f_i on input x:

$< i, n, x, j > \xrightarrow{Oracle\ O}$ the j-th bit of $f_i(x)$, where f_i is the i-th function $f_i : \{0,1\}^n \rightarrow \{0,1\}^n$.

Now, that we have covered all the necessary background we move to the *central part of our project* which focuses on the Generalised Encryption Scheme and the Complexity Theory results related to it.

4. Generalised Encryption Scheme

Consider the following situation: a client wants to compute a hard function. Because he does not have the necessary computational resources to do it by himself, he wants to use an untrusted server which is capable of solving difficult problems. In this case, the client needs to somehow encrypt his input to the problem, such that the server cannot find out any kind of information about it.

This is the purpose of *Generalised Encryption Scheme(GES)* [16].

DEFINITION 4: *GES* is an *encryption scheme framework* solving the *delegated computation task* [14] and can be described by the following sequence of steps:

S1. The client applies the encryption primitive on his input and sends the resulting cyphertext to the server.

S2. The sever performs the computation on the ciphered data and returns the outcome back to the client.

S3. The client applies the decryption procedure and in this way he obtains the result of his desired computation.

OBS 1: A first observation is that this encryption scheme must satisfy the property that by applying the hard function on the encrypted input and then decrypting the resulting outcome, will give us the result of applying the function on the cleartext input.

Basically, this scheme would allow any user with limited computational capabilities to solve any problem by exploiting a powerful server, while protecting the privacy of his input. Moreover, this scheme guarantees *information-theoretic security* [9]. This means that no matter how powerful a malicious server might be and no matter what actions he might try (or even if a third party could have access to the encrypted input), the security of this protocol cannot be broken [12].

OBS 2: Achieving *information-theoretic security* implies that the encryption must not be based on computationally infeasible problems. If we assume that the server would have unlimited power, then he would be able to solve those problems [11] and obtain the plaintext input of the client.

Therefore, by taking into consideration its security premises, this is an extremely important problem, as a *GES* protocol would bring major consequences in *cloud computing* and in principle it can guarantee *secure processes* over the Internet [4, 14].

Furthermore, this scheme has significant consequences in the field of *Complexity Theory* [16, 17]. We will next derive relations between the complexity of solving a problem f and the complexity of encrypting f in the *GES* scenario. Specifically, we will analyse the implications for a function which belongs to a complexity class C, to admit a *GES* scheme.

Below, we are going to give a **precise description** of the *GES* framework.

Suppose a client A wants to compute the value of $\boldsymbol{f(x)}$, where $f : M \to N$ and $x \in M$. Then, A must choose a key $\boldsymbol{k} \in \mathcal{K}$ which will be used along the entire evolution of the protocol. The encryption primitive $\boldsymbol{E}$ takes as input x and the key k and returns the cyphertext y, $E : M \times \mathcal{K} \to M$. The obtained string y is then sent to the server B which applies the function f and returns $f(y)$ to A.

The decryption primitive $\boldsymbol{D}$ takes $f(y)$, the input x and the key k and returns the *desired result*:

$D : N \times M \times \mathcal{K} \to M$ such that $D(f(y), x, k) = f(x)$, where $y = E(x)$

OBS 3: In terms of *privacy*, the scheme does not necessarily need to hide everything about input x. For instance, the client may only want to hide the first half of the bits of the input.

Therefore, we consider a function L which takes as input a string from M and defines how much does the scheme *leak* about an input x (what is the maximum allowed information about an input $x \in M$ that can be inferred by the server B) [16]. From now on, all the results obtained for specific classes of problems solvable in the *GES* scenario will be made in relation to the function L.

DEFINITION 5: *GES* is defined in a broader context as an *interactive protocol* in which A and B will exchange $\boldsymbol{m}$ pairs of messages in the following way:

- At the beginning, A uses a randomized key-generation procedure $\boldsymbol{K}()$ which receives x and returns a key $k \in \mathcal{K}$ which will be used in all the m sessions of communication;
- Then, A calls the encryption function E on the pair (x, k) and sends as a first message $\boldsymbol{a_1} = E(x, k)$. Accordingly, B responses back to A with a message $\boldsymbol{b_1}$;
- For the next round, A's encryption procedure will also consider B's response b_1 and will generate the new message $\boldsymbol{a_2} = E(x, k, [b_1])$. Hence, at step i, A's message will be based on

all previous message received from B: $\boldsymbol{a_i} = E(x, k, [b_1, b_2, \dots, b_{i-1}])$, which is thereafter sent to B;

- B chooses his messages, $(b_1, b_2, \dots, b_m)$, in such a way that after the last round of communication, when A finally applies the decryption function which takes all m messages received from B, the result $\boldsymbol{r} = D(x, k, [b_1, b_2, \dots, b_m])$ is equal to $f(x)$ with *probability* $\frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{|x|^c}$, where c is a constant.

We can observe that B's responses b_i do not even have to be elements from the codomain of f; they can be chosen from any distribution known by B, as long as they satisfy the *condition* mentioned above.

OBS 4: Considering that the client A has only limited computational power we must add the restraints that the number of rounds, m, must be polynomial in $|x|$ and that the procedures of encryption $E(\,)$, decryption $D(\,)$ and key-generation $k(\,)$ must run in polynomial time. Particularly, as $k(\,)$ is a randomized algorithm, $k(\,) \in ZPP$ [20].

We will now present some *examples of hard problems* [7, 17] and show how they are *solved* using the *GES* protocol, namely we describe how the encryption, decryption and key-generation functions are defined for each of these problems [16].

Example 1: Suppose f is the *discrete logarithm problem* (which is known to be in NP and BQP). In this problem we are given a prime number p, a generator g of the group $\mathbb{Z}_p$ and an element u of $\mathbb{Z}_p$, $u \in \{0, 1, \dots, p-1\}$. The problem asks us to determine the unique number $e \in \mathbb{Z}_p$ such that $g^e \equiv u \bmod p$.

Therefore, the input x consists of the tuple (p, g, u). We set the key-generation procedure to return a random number $k \in \mathbb{Z}_p$ and choose the encryption procedure to act in the following way:

$E((p, g, u), k) = (p, g, v)$, where $v = \left(u \cdot g^k\right) \bmod p$.

Then, B receives (p, g, v) and solves the discrete logarithm problem for this input. This means that he determines the value $e' \in \mathbb{Z}_p$ such that $g^{e'} \equiv v \bmod p$.

Now, we need to find a decryption procedure which given e' and k returns the desired result e.

This decryption function is $D(e', k) = (e' - k) \bmod p$.

We can show that this procedure does return the correct solution by verifying that $g^e \equiv u \bmod p$, where $e = (e' - k) \bmod p$.

$g^e = g^{(e'-k) \bmod p} \equiv g^{e'} \cdot \left(g^k\right)^{-1} \bmod p$. But, we know that $g^{e'} \equiv v \bmod p$, so we obtain:

$g^e \equiv v \cdot \left(g^k\right)^{-1} \bmod p = u \bmod p.$

In this way, we have defined a *GES* scheme for the *discrete logarithm problem* which *leaks* the values of p and g (as p and g are sent directly in plaintext to B). $\square$

Example 2: Consider f to be the *primitive root problem*. This problem receives as input a prime number p and an integer g and checks if the number g is a *primitive root modulo* p. This is equivalent to verifying that for every integer a coprime with p there exists an integer k such that $g^k \equiv a \bmod p$.

The key-generation procedure returns an integer $k \in \{1, \dots, p-1\}$ such that k is coprime with $p-1$.

The encryption procedure is defined as: $E\big((p,g),k\big) = (p,g')$, where $g' = g^k \bmod p$.

Then, B receives (p,g') and returns 1 if g' is a primitive root modulo p and 0 otherwise. Once A receives this outcome the decryption function just needs to copy this answer due to the next relation:

g' is primitive root mod $p \Leftrightarrow g$ is a primitive root mod p.

Thus, we found a *GES* scheme for the *primitive root problem* leaking p and the order of $g \bmod p$. $\square$

> ***Example 3:*** Suppose f is the *quadratic residuosity problem*. In this problem we are given a number n, such that $n = p \cdot q$, with p and q prime numbers and $u \in \mathbb{Z}$. We need to check if there exists a number $b \in \mathbb{Z}$, such that $u \equiv b^2 \bmod n$.

The key generation procedure returns a key k consisting of a pair of numbers (c, d), where $c \in \mathbb{Z}_n$ and $d \in \{0,1\}$, both numbers selected at random.

The encryption primitive is constructed as: $E\big((n,u),k\big) = (n,v)$, where $v = \big(u \cdot c^2 \cdot (-1)^d\big) \bmod n$

Therefore, B gets (n, v) and determines whether v is a quadratic residue modulo n and returns to A the answer r.

The decryption function used by A returns r if d was 0, and $1-r$ if d was 1.

As seen, this *GES* scheme built for the *quadratic residuosity* problem leaks only the number n. $\square$

We now move on to analyse the connections between the complexity class of a problem f and how hard is solving f using *GES*. The first important result is the following [16]:

THEOREM 8. *If f belongs to the class ZPP, then f can be solved using a GES scheme that hides everything about the input*.

Proof. If $f \in ZPP$ we can define the following *GES* scheme:

We select a random number z from the domain of f. Then, we can define the 3 primitives of the client in this way:

$K(x) = f(x)$; $E(x,k) = z$; $D(r,k,x) = k$;

Basically, because the problem is in ZPP it can be directly computed by the client. Therefore, the key-generation procedure can be used to determine the desired result, $f(x)$. The encryption just returns a random element from the domain, making sure that the server cannot infer any information about x, not even its size. $\square$

THEOREM 9. *If f can be solved using a GES scheme that hides even the length of the input, then f must be in the class ZPP.*

Proof. Given a function f computed in a *GES* scheme which hides everything about the input, including its length, we show that we can construct an algorithm in ZPP that can solve f.

First, we select a fixed input z from the domain of f.

We know that the encryption procedure runs in polynomial time (OBS 4), therefore there ∃ a polynomial p such that ∀ $k \in \mathcal{K}$ valid for the input z, we have that $E(z,k) = y_k$ and $|y_k| \le p(|z|)$.

We define the set $\mathcal{Y} = \{y \in Domain(f) |\, |y| \le p(|z|)\}$. Definitely, we have that $\mathcal{Y}$ is finite and that any possible encryption of z must be in this set.

Take x any input for the function f. Because the protocol also needs to hide the size of x, then its encryption y must lie in $\mathcal{Y}$. Otherwise, if $|y| > p(|z|)$, then B could infer some information about the length of x (for instance that $|x|$ is different from $|z|$).

Now, as we know that the number of possible outcomes of E is finite (less or equal than $card(\mathcal{Y})$) we can perform an *initial computation* [21](before the actual running of the algorithm), where we store in a table T the values of $f(y)$ for any $y \in \mathcal{Y}$.

Then, the ZPP algorithm for solving f is the following:

```
Algorithm $f(x)$ {
        Draw a key $k$ from $\mathcal{K}$;   // this is the probabilistic part of the algorithm, as we might
                                  // select a key $k$ which is not valid for $x$, so there is a probability that
                                  // the final answer of this algorithm is "DO NOT KNOW"
        Compute $y = E(x)$;
        Search the value of $f(y)$ in table $T$;
        Apply decryption $D(f(y), k, x) = f(x)$;
}
```

□

These 2 results show us that if *GES* can hide the length of the input, then the scheme *can hide everything* about it, but if we do not want **to leak anything about the input**, then the **function must be in ZPP** [16, 17].

Because this privacy constraint of hiding everything about the client's data might be unnecessary and because it prevents any *hard* functions from being computed in the *GES* scenario, we will instead consider *GES* schemes that *leak the size of the input*, $L(x) = |x|$.

Based on the proofs for the ZPP functions, we might be tempted to believe that the difficulty of encrypting functions using *GES,* grows with the difficulty of solving them. However, this is not the case, as illustrated by the next example.

Example 4. We construct a function f in the following manner: for each possible length n of an input x we select a Turing Machine $\mathcal{M}_n$ whose encoding is a string s, such that $|s| = |x| = n$.

Then f is defined as:

$$f(x) = \begin{cases} 1, & if\ \mathcal{M}_{|x|}\ halts\ for\ all\ possible\ inputs \\ 0, & otherwise \end{cases}$$

Determining whether a TM halts on all inputs is known to be an *undecidable problem* [34](there does not exist any algorithm which can solve this decision problem). By the construction of f, we obtain that f is also *undecidable*.

However, there exists a *GES* scheme which could solve f and which leaks only the length of the input.

The encryption function picks any random number y from the domain of f such that $|x| = |y|$.

Then, B would solve the problem for the input y, and the corresponding answer, $f(y)$, will be equal to the outcome for our input x (because $f(x) = f(y)$ for any y such that $|y| = |x|$).

The presented *counterexample* indicates that if $f(x)$ can be written as a *function of the length* $|x|$ ($\exists$ function h such that $f(x) = h(|x|)$), then f can be encrypted using a *GES* leaking $L(x) = |x|$. Therefore, in this case, no matter how hard is the function h, f can be solved using the *GES* protocol. $\square$

OBS 5: This conclusion reveals the link between classes of problems solvable in the *GES* protocol and classes of problems solvable using *advice functions*.

We remind that the advice function is an *external information* helping a machine to solve different problems, which does not depend on the value of the input, but only on its length [28]. For instance, $P/poly$ is defined as the class of languages:

$C = \{x \mid (x, h(|x|)) \in A, A \in P \text{ and } h(n) \leq poly(n) \; \forall \, n\}$.

Then, we observe that for every function f such that $f(x) = h(|x|)$, we have that f belongs to the class $P/poly$ [24]. This is because for any input x, the advice could directly give the value $h(|x|)$, thus solving f.

Using the result of Karp [20] which says that no $NP - hard$ problems can be solved by a $P/poly$ machine, we obtain that *if f only depends on the length of its inputs* and f is $NP - hard$, then f *cannot be solved* using a *GES* protocol which leaks $L(x) = |x|$.

Now, we want to move to the general case for any type of function f and study the relation between encrypting f and computing f with the aid of an external advice.

We have the following crucial result of Abadi [16]:

MAIN THEOREM 1: **If a function f can be solved using a *GES* scheme which leaks the size of the input, then: $f \in NP/poly \cap coNP/poly$.**

Basic case. We begin our proof from a simpler scenario of *GES*:

- 1 round of communication between A and B in which A sends $y = E(x, k)$, B computes $f(y)$ and returns it to A and finally, A decrypts $f(y)$ and obtains $f(x)$.

We can define the following algorithm in $NP/poly$ which computes the function f.

For any possible length n we consider the input string 0^n and choose a valid key k_n for 0^n.

The advice can be any kind of information which could help an algorithm solve f, as long as it is the same piece of data for all inputs of the same length. Therefore, for the length n we define the advice as the pair (u_n, v_n), where $u_n = E(0^n, k_n)$ and $v_n = f(u_n)$.

Then, the nondeterministic algorithm using this advice is the following [16]:

```
$f(x)${
        from the advice function we receive the pair $(u_n, v_n)$ where $n = |x|$;
        $k$ = choice from the space of keys, $\mathcal{K}$, such that $E(x, k) = u_n$; // the nondeterministic step
        apply the decryption function $D(v_n, k, x) = f(x)$ ;
}
```

The *correctness* of this algorithm is guaranteed by the fact that we can find a key k such that $E(x, k)$ = $E(0^n, k_n)$. This is due to the fact that since the encryption function leaks only the input size, then for any inputs x_1, x_2 such that $|x_1| = |x_2|$, there exist 2 keys k_1, k_2 such that $E(x_1, k_1) = E(x_2, k_2)$.

$\square$

Now, we can advance to the **general *GES* scenario**. The main issue in this case is not the fact that there are multiple rounds of interaction between A and B, but that the final result A obtains is $f(x)$ only with some probability p = $\left(\frac{1}{2} + \frac{1}{|x|^c}\right)$. Showing that f can be solved by an NP algorithm using advice, implies we must *always* get the correct answer $f(x)$ [17].

DEFINITION 5.1: We define a *transcript* t as the set of pairs of messages exchanged between A and B during the *GES* protocol, $t = [(a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m)]$.

The final result derived by A is based on the input x, a key k (chosen by A at the beginning of the protocol) and the transcript t denoting the actual communication between A and B. Therefore, this outcome is a function of x, k and t, $out = F(x, k, t)$ and from the definition of *GES* we know that $out = f(x)$ with probability p.

We can associate for every input x, a *distribution* of possible transcripts t. As the encryption hides everything but the length of x, this probability distribution of transcripts is equal for all inputs of the same size. Therefore, for every possible length n of the input, we have a corresponding random variable $\boldsymbol{T_n}$ representing the transcript distribution [34].

Below, we are going to make some additional notations which will help us during the proof.

DEFINITION 5.2: We call a transcript t ***suitable*** for a given input x and a key k if there is a strictly greater than 0 probability that the actual communication between A and B, is equal to t, given x and k.

DEFINITION 5.3: $\boldsymbol{k_{x,t}}$ is the set of keys k_i such that the transcript t is *suitable* for x and k_i. Verifying that a key belongs to this set can be done in polynomial time: we fix B's responses exactly as they appear in transcript t, then compute A's behaviour given the key k, input x and B's responses and lastly, verify if $A's$ messages match the set (a_1, …, a_m) from t.

DEFINITION 5.4: Thus, we say that a key k is ***good*** for an input x and a transcript t, if $k \in k_{x,t}$ and the final result obtained by A, $out = F(x, k, t)$, is equal to $f(x)$; $\boldsymbol{kg_{x,t}}$ is the set of *good* keys, while the rest of the keys, which we call *wrong* keys, are in the set $\boldsymbol{kw_{x,t}}$.

Looking back at the previous proof for the *Basic case* (with 1 round of communication between A and B), we notice that the advice consisted of the pair $(y, f(y))$, which can be seen as a transcript with only 2 messages. In the general case of *GES,* we intend to determine a set S = $\{t_1, \ldots, t_l\}$ containing a polynomial number of *well-chosen transcripts* such that S will be used as the advice to a nondeterministic machine in order to compute $f(x)$.

While in the ***Basic case***, the nondeterministic step was the choice of k which led us to the same encryption u_n for all inputs of size n, in the general *GES* scenario the nondeterministic computation will be represented by some choices of keys k from the sets of *good keys* kg_{x,t_i}, where $t_i \in S$.

Suppose the advice S would consist of only one transcript t. Then, we must be able to find a key k_i such that for any input x, k_i is *good* for (x, t). But there might be the case that using the same key k_i for any x, would not always generate the same transcript t [32]. Therefore, we need to consider more than just one transcript for S.

Now we must determine which transcripts should be chosen as advice.

The key used by the A along the entire protocol is the outcome of the random key-generation procedure $k(\,)$ which runs in polynomial time. As $k(\,)$ is a randomized algorithm ($k(\,) \in ZPP$), it uses an additional tape consisting of an infinite row of random bits. Thus, $k(\,)$ can be perceived as a deterministic procedure which receives 2 inputs: x and w, where w is comprised of random bits from that extra tape [17, 20].

We examine the case when this key-generation primitive $k(\,)$ has only a finite number of random bits which it can access and consider this number to be polynomial in n, $\boldsymbol{r(n)}$.

DEFINITION 5.5: For every choice of the polynomial $\boldsymbol{r}$, we obtain a new ***reduced GES*** $\boldsymbol{G_r}$ [16], created from the original *GES* G. Consider $k_r(\,)$ the new key generation function that uses exactly $r(n)$ random bits for every x of size n. Then, the *reduced GES* G_r is defined in the following way:

Given an input x, $k_r(\,)$ produces a key k'. Then, the bits of k' will be used as a tape for the key-generation procedure of G, $k(\,)$. After $k(\,)$ returns a key k, G_r just replicates the behaviour of G. Essentially, we do not alter the algorithm of $k(\,)$, we just restrict the size of the domain for $k(\,)$. Therefore, G_r should give us the same final answer we would have obtained if we used G.

However, the key-generation procedure of G $k(\,)$, might require more than $r(n)$ random bits and in this case G_r would stop and we would not have any transcript at all.

We describe the probability of this event happening as:

DEFINITION 5.6. $break_r(x,t) = P(k(\,)\ requires\ more\ than\ r(n)\ bits \mid T_n = t\ and\ X = x)$.

DEFINITION 5.7. We denote $Break_r(x)$ as the *expected value* of $break_r(x,t)$ over the distribution of t Hence, $Break_r(x)$ defines the probability that the *reduced* G_r cannot deliver any transcript for an input x, if we only allow $r(|x|)$ random bits to $k(\,)$.

To analyse how appropriate is a transcript to be used as advice, we need to consider the following indicators:

$$\boldsymbol{fail(x,t)} = P(G\ obtains\ a\ result\ different\ from\ f(x) \mid T_n = t\ and\ X = x)$$

$$\begin{aligned}\boldsymbol{fail_r(x,t)} &= P(G\ obtains\ a\ result\ different\ from\ f(x) \mid T_n = t\ and\ X = x\ and \\ &\qquad k(\,)\ requires\ no\ more\ than\ r(n)\ bits) \\ &= P(G_r\ obtains\ a\ result\ different\ from\ f(x) \mid T_n = t\ and\ X = x)\end{aligned}$$

We observe that $fail_r(x,t) = 1$ if G returns the correct answer only when $k(\,)$ needs more than $r(n)$ bits. (1)

We use the conditional probability relation [34]: $P(U \mid V,W) = \frac{P(U,W,V)}{P(W,V)} = \frac{P(U,W \mid V)\cdot P(V)}{P(W|V)\cdot P(V)} = \frac{P(U,W|V)}{P(W|V)}$.

If we make the substitutions: U = "G obtains the wrong result", V = "$T_n = t \ and \ X = x$" and W = "$k()$ uses at most $r(n)$ bits", we obtain that: $fail_r(x,t) = P(U \mid V,W)$, $break_r(x,t) = 1 - P(W \mid V)$ and $fail(x,t) = P(U \mid V)$. Therefore, we have:

$fail_r(x,t) = \frac{P(U,W \mid V)}{1 - break_r(x,t)} \le \frac{fail(x,t)}{1 - break_r(x,t)}$ (we used (1) for the inequality) (2)

The choice for a transcript t will be made using the following evaluation:

DEFINITION 5.8: We say a transcript t is ***reliable*** *for an input* x if it satisfies the conditions:

$i)\ fail_r(x,t) < \frac{1}{2^n}$ $\qquad ii)\ break_r(x,t) < \frac{1}{10}$;

The 2 values $\frac{1}{2^n}$ and $\frac{1}{10}$ just indicate that the 2 probabilities representing failures of the GES scheme, are close to 0 [21, 34].

We can rewrite these conditions using inequality (2) and thus, we obtain that t is ***reliable*** when:

$$\begin{cases} \frac{fail(x,t)}{1 - break_r(x,t)} < \frac{1}{2^n} \\ break_r(x,t) < \frac{1}{10} \end{cases} \Leftrightarrow \begin{cases} fail(x,t) < \frac{9}{10 \cdot 2^n} \\ break_r(x,t) < \frac{1}{10} \end{cases}$$

The reason why we consider these *reduced* versions of *GES* (where we restrict the number of random bits available for $k(\,)$) is because they allow us to achieve new *GES* schemes with a lower failure rate, as we shall later see. But before that, we need to make some notations for the *reduced* G_r, exactly as we had for the general *GES*: $\boldsymbol{k^r_{x,t}}$ = $\{k \mid t$ is *suitable* for x and $k\}$, $\boldsymbol{kg^r_{x,t}}$ - the *good* keys for an input x and transcript t and $\boldsymbol{kw^r_{x,t}}$ - the *wrong* keys (which do not lead to a correct answer). Now, we can use the properties of *reduced GES* to formulate the following relations:

We remind that $k_r(\,)$ generates equally probable keys all of length $r(|x|)$.

- $P(T_n = t \mid X = x)$ can be computed as the number of keys k such that from (k,x) we obtain the transcript t, divided by the total number of keys k generated by $k_r(\,)$:
  $P(T_n = t \mid X = x) = \frac{|k^r_{x,t}|}{2^{r(|x|)}}$
- 1 - $break_r(x,t) = P(k(\,)\ uses\ less\ than\ r(n)\ bits \mid X = x\ and\ T_n = t)$ which can be computed as the number of keys k of size less than $r(n)$ such that (k,x) can generate t, divided by the total number of keys which along with x can produce t $\Rightarrow$ $1 - break_r(x,t) = \frac{|k^r_{x,t}|}{|k_{x,t}|}$
- $fail_r(x,t)$ – can be computed as the ratio between the number of *wrong* keys and the total number of keys which along with x can generate t $\Rightarrow$ $fail_r(x,t) = \frac{|kw^r_{x,t}|}{|k^r_{x,t}|}$ (3)

Using these definitions and the fact that G_r hides everything but the input length, we obtain that for any 2 inputs x and x' such that $|x| = |x'|$, we have the following property:

$$\frac{P(T_n=t \mid X=x)}{P(T_n=t \mid X=x')} = \frac{|k^r_{x,t}|}{|k^r_{x',t}|} = \frac{1 - break_r(x,t)}{1 - break_r(x',t)} \qquad (4)$$

What we want is to ensure that we can reduce the probability that a *GES* gives an incorrect answer for $f(x)$, to a minimum, as close to 0 as possible, irrespective of the transcript between A and B.

We can show the following important result [16]:

THEOREM 10.1 : From any *GES G* we can construct a *reduced GES* G_r, such that G_r satisfies the following 2 conditions:

$$1)\ E(fail(x,T_n)) < \frac{1}{2^{d\cdot n}} \quad \text{and} \quad 2)\ Break_r(x) < \frac{1}{n^d} \qquad \forall d \text{ constant}$$

Proof. Given the *GES* definition, we know that for any *GES,* there exists a constant c such that $E(fail(x,T_n)) < \frac{1}{2^{c\cdot n}}$. Thus, we only need to show that the 2 properties hold for any constant $d > c$.

Furthermore, because $k(\,)$ must run in expected polynomial time [22], there exists a polynomial q such that for any input x the expected number of bits required by $k(\,)$ is less than the value of $q(|x|)$.

Then, we consider the *reduced GES* G_z, where the polynomial z is defined as $z(n) = n^d \cdot q(n)$.

If we denote by Z the random variable [28] indicating the number of bits required by the function $k(\,)$, we obtain that the quantity $Break_z(x)$ associated with the scheme G_z is equal to the probability that $k(\,)$ requires more than $z(n)$ bits, which is $P(Z > z(n))$.

We have the following probability theory known result [34]: for any random positive variable X and for any positive real number t: $P(X > t) \leq \frac{E(X)}{t}$. (5)

Writing this relation for our case, we have that: $Break_z(x) = P\big(Z > z(n)\big) \leq \frac{E(Z(x))}{z(n)}$. But, by the definition of q, $E\big(Z(x)\big) = q(n)$, so we reach the inequality: $Break_z(x) \leq \frac{q(n)}{q(n)\cdot n^d} = \frac{1}{n^d}$.

The probability that G_z outputs a *wrong outcome* is by definition $\frac{1}{2} - \frac{1}{n^c}$ [16]. Therefore, we obtain $P(G_z\ breaks\ or\ wrong\ output) = \frac{1}{2} - \frac{1}{n^c} + \frac{1}{n^d}$ and because $d > c$:

$P(G_z\ breaks\ or\ wrong\ outputs\ ) < \frac{1}{2}$.

Hence, if we simulate the *reduced GES* G_z $\boldsymbol{m = d \cdot n}$ **times** and choose the answer which appears with the highest frequency in these runs, then this probability becomes less than $\frac{1}{2} \cdot \ldots \cdot \frac{1}{2} = \frac{1}{2^{dn}}$.

Using G_s, we construct a new *reduced GES* G_r which respects the 2 conditions of THEOREM 10.1.

We set r such that $r(n) = z(n) \cdot m\ \forall n$.

The key-generation primitive $k_r(\,)$ runs the key-generation procedure of G_z m times, thus resulting a set of m keys: $k_1, \ldots, k_m$. For each of these keys we essentially have a different run of a *GES* protocol, therefore, we obtain m different transcripts at the end of which by applying the decryption algorithm of A, will result m final outcomes (as candidates for $f(x)$). Then, we can choose as our solution for $f(x)$ the outcome with the highest frequency amongst these m results [30].

This new defined scheme G_r respects the settings of a *GES*, so now we need to check if it really satisfies the 2 conditions imposed by THEOREM 10.1. We have already shown that if we run the G_z scheme $m = d \cdot n$ times we obtain that $P(fail(x,T_n)) < \frac{1}{2^{d\cdot n}}$, so condition 1) is achieved.

Additionally, the *break* probability of G_r is equal to the probability that all m runs of G_s *break* and we saw that this probability is less than $\frac{1}{n^d}$. This concludes that condition 2) is also true, so we have constructed a GES G_r originating from G, which satisfies both 1) and 2). $\square$

As G was any arbitrary *GES*, this means that for every *GES* we can use this algorithm to obtain a new *GES* with the properties that: $\boldsymbol{a}$) $E(fail(x, T_n)) < \frac{1}{2^{d\cdot n}}$ and $\boldsymbol{b}$) $Break_r(x) < \frac{1}{n^d}$ for any constant d.

Therefore, we can assume that from now one we will work with a GES G with these 2 properties.

Now from this type of *GES G* we show the following theorem [16]:

THEOREM 10.2: There exists a fixed value e and a number N, such that $\forall n > N$ we can find a set S_n consisting of n^e transcripts, where more than a half of the transcripts are *reliable* for any input x of length n.

Proof. We can intuit that from this set S_n we would derive our advice, as most of the transcripts in S_n are *reliable* for all inputs. We remind that a transcript t is *reliable* (DEFINITION 5.7) when $fail(x,t) < \frac{9}{10\cdot 2^n}$ and $break(x,t) < \frac{1}{10}$.

Let's first consider any possible transcript we could have for G. We randomly choose any such transcript t and analyse the probability that t is not *reliable*.

From property $\boldsymbol{b}$) of G we know that $E(Break(x)) < \frac{1}{n^d}$ $\forall d$ .We can choose $d = 2$, so we obtain: $E(Break(x)) < \frac{1}{1000}$ for any $n > N = \sqrt{1000}$.

Using the inequality (5) we have that $P\left(break(x,t) > \frac{1}{10}\right) \leq \frac{E(Break_r(x))}{\frac{1}{10}} \leq \frac{1}{100}$.

Similarly, from property $\boldsymbol{a}$) of G, we get that $E(fail(x,T_n)) < \frac{1}{2^{d\cdot n}}$ $\forall d$. When $d = 2$, we obtain: $E(fail(x,T_n)) < \frac{1}{2^{2n}}$ $\forall n$.

Again, by applying inequality (5) we reach that $P\left(fail(x,t) > \frac{9}{10\cdot 2^n}\right) \leq \frac{E(fail(x,t))}{\frac{9}{10\cdot 2^n}} \leq \frac{10}{9\cdot 2^n} \leq \frac{1}{100}$

This means that the probability that t is *not reliable* for x is less than $\frac{1}{100} + \frac{1}{100}$, so the actual probability that t is *reliable* for any x, $P(t\ is\ reliable) > 1 - \frac{2}{100} = \frac{98}{100}$

Now, if we have a set of k independent transcripts t_i, we would like to see what is the probability Pr that less than half of them are *reliable* for any input x. (by looking at what THEOREM 10.2 requests**,** we want this probability to be as close to 0 as possible).

We can view Pr as the discrete probability of the number of acceptances in a row of k independent *yes-no* trials (*yes* = t_i is *reliable*, no = t_i is *unreliable*), each being accepted a probability p (in our case $p = \frac{98}{100}$). In this way, we modelled the probability Pr as a *binomial distribution* [34], so we have that:

$Pr = P\left(number\ of\ reliable\ transcripts\ for\ x < \frac{k}{2}\right) = F(\frac{k}{2}, k, p)$, where F is the *cumulative distribution function* for the *binomial distribution* [28].

Using *Chernoff's inequality* [21, 34] stating that: $F(m,n,p) \leq exp(-\frac{1}{2p} \cdot \frac{(np-m)^2}{n})$, we obtain:

$$P\left(number\ of\ reliable\ transcripts\ for\ x < \frac{k}{2}\right) < \exp\left(-\frac{1}{2 \cdot \frac{98}{100}} \cdot \frac{\left(\frac{98k}{100} - \frac{k}{2}\right)^2}{k}\right) = \frac{1}{2^{O(k)}}$$

Therefore, for a set S of $k = n^e$ transcripts, $e \geq 1$ constant, this probability is less than $\frac{1}{2^{O(n^e)}} < \frac{1}{2^n}$. But as there are 2^n possible inputs of length n, when we cumulate this probability for every input we reach that : $P\left(number\ of\ reliable\ transcripts\ for\ all\ x < \frac{|S|}{2}\right) < 2^n \cdot \frac{1}{2^n} = 1.$

Then, since the probability is strictly less than 1, this means that there must exist a set S_n satisfying the conditions mentioned in THEOREM 10.2. $\square$

OBS 6: We make the next important observation: for any *reliable* transcript t from S_n we have an associated set of keys k which could produce t, namely $k(x,t)$. But, in order to reach the correct result $f(x)$, we need to **make a distinction** between the subsets of ***good*** **keys** (the ones that lead to $f(x)$) and the set of ***wrong*** **keys** (the ones which lead to an incorrect answer). Then, we could use the *good* keys corresponding to the *reliable* transcript t, as part of the ***external advice*** to solve $f(x)$ in NP.

This separation between *good* and *wrong* keys is achieved through the use of ***Universal Hashing*** [35].

DEFINITION 5.9: *Universal hashing* can be described as follows: we have H, a set of independent hash functions that map elements from any set S to the set $T = \{0, \dots, m-1\}$. H has the important property that by randomly choosing any function h from H, we have a small number of *collisions*: $P\big(h(a) = h(b)\big) < \frac{1}{m}$ $\forall a, b \in S$.

In our case we use the universal hashing H to map *keys* of G to the set $T = \{0, \dots, m-1\}$.

Consequently, using the *collision property* of H [35], we have that for a subset of p keys, $\{k_1, \dots, k_p\}$ and $h \in H$, the probability that all p keys are mapped to exactly the same value is:

$$P\left(h(k_1) = \cdots = h(k_p)\right) = \underbrace{\frac{1}{m} \cdot \frac{1}{m} \cdot \ldots \cdot \frac{1}{m}}_{p-1} = \frac{1}{m^{p-1}}.$$

DEFINITION 5.10: We name a set of keys $L = \{k_1, \dots, k_p\}$ for which we have $h(k_1) = \dots = h(k_p)$, a (h,p)-*collision set*.

The separation mentioned before, between *good* and *wrong* keys is based on the following result [16]:

THEOREM 10.3 For any transcript t there exist 2 numbers m and p and a function $h \in H$ such that:

(A) The set of *good* keys $kg_{x,t}$ includes a (h,p)-*collision set,* for any x such that t is *reliable* for x;
(B) The set of *wrong* keys $kw_{x,t}$ cannot include any (h,p)-*collision sets,* for any x such that t is *reliable* for x;

Proof. We notice that if the number of *good* keys, $|kg_{x,t}|$ is larger than $p \cdot m$, then any hash function h must map at least $p \cdot m$ keys to the m elements of T [34]. In this case, there needs to exist at least p keys such that h maps them to the same element of T, which would result in a (h,p)-*collision set.*

To satisfy the (B) condition we must analyse the probability that for any randomly chosen $h \in H$, h can produce a (h,p) – collision set. After that, we determine values for m and p such that this probability is strictly less than 1.

Consider the class of all sets of *wrong* keys $kw_{x_i,t}$ corresponding to a x_i such that t is *reliable* for x_i. We set U to be the maximum of the cardinalities of these sets.

For a given x_i, we can choose a subset of p keys from a maximum number of U keys in $\binom{U}{p}$ ways.

The probability that any of these subsets could form a (h,p)-*collision set* becomes $\binom{U}{p} \cdot \frac{1}{m^{p-1}}$ (by the definition of H).

As the total number of inputs x_i, such that t is *reliable* for them, is at most 2^n ,we obtain that the probability of having a (h,p) – collision set containing only *wrong* keys from $kw_{x_i,t}$ is less than

$\binom{U}{p} \cdot \frac{1}{m^{p-1}} \cdot 2^n$.

This means that we need to find values for p and m such that this quantity is strictly less than 1.

Using *Stirling's formula* [21] we have that $M = \binom{U}{p} \cdot \frac{1}{m^{p-1}} \cdot 2^n < \left(\frac{eU}{p}\right)^p \cdot \frac{1}{m^{p-1}} \cdot 2^n \ = m \cdot 2^n \cdot \left(\frac{eU}{pm}\right)^p$

By setting the value of m such that $eU < m < 2eU$, we obtain that $M < 2^n \cdot 2eU \cdot \frac{m^p}{p^p \cdot m^p} = \frac{2^{n+1}eU}{p^p}$.

Because every key k must be of polynomial size, there exists a constant q such that $k < n^q$. Then, since U is a subset of all the possible keys, we have $U < 2^{n^q}$.

Therefore, $M < \frac{e \cdot 2^{n+1+n^q}}{p^p}$. Now, we just need to set the value of p such that M is always strictly less than 1:

$\frac{e \cdot 2^{n+1+n^q}}{p^p} < 1 \Leftrightarrow 2^{n^q+n+1+elog(e)-plog(p)} < 1$. So, by setting $p = n^{q+1}$we have satisfied condition (B).

We observed before that in order to meet condition (A) we must make sure that $|kg_{x,t}| > p \cdot m$.

To verify this we use the relation (4) which specifies that for any 2 inputs x, x' we have:

$\frac{|k_{x,t}|}{|k_{x',t}|} = \frac{1 - break(x,t)}{1 - break(x',t)}$.

We choose x and x' in the following way:

- x is the input x_i such that t is *reliable* for x_i and the corresponding set of keys $k_{x_i,t}$, has the property that: $\left|k_{x_i,t}\right| = \min_{x_j} \left|k_{x_j,t}\right|$
- x' is the input x_i such that t is *reliable* for x_i and the corresponding set of keys $k_{x_i,t}$, has the property that: $\left|k_{x_i,t}\right| = \max_{x_j} \left|k_{x_j,t}\right|$

Then, relation (4) becomes: $\frac{\min_{x_j}\left|k_{x_j,t}\right|}{\max_{x_j}\left|k_{x_j,t}\right|} = \frac{1 - break(x,t)}{1 - break(x',t)}$, which is equivalent to:

$$\min_{x_j} \left|k_{x_j,t}\right| = \frac{\max_{x_j}\left|k_{x_j,t}\right| \cdot (1 - break(x,t))}{1 - break(x',t)} \geq \max_{x_j} \left|k_{x_j,t}\right| \cdot \left(1 - break(x,t)\right)$$

Additionally, we know that for any transcript t of G we have that $break(x,t) < \frac{1}{10}$, thus we obtain: $\min_{x_j} \left|k_{x_j,t}\right| > \frac{9 \cdot \max_{x_j}\left|k_{x_j,t}\right|}{10}$. By making the notation $V = \max_{x_j} \left|k_{x_j,t}\right|$, the inequality becomes: $\min_{x_j} \left|k_{x_j,t}\right| > \frac{9 \cdot V}{10}$.

Now, the quantity we are interested in, $|kg_{x,t}|$ is the difference between the total number of keys and the number of *wrong* keys:

$$|kg_{x,t}| = |k_{x,t}| - |kw_{x,t}| \geq \min_{x_j} |k_{x_j,t}| - \max_{x_j} |kw_{x_j,t}| = \min_{x_j} |k_{x_j,t}| - U > \frac{9 \cdot V}{10} - U$$

Using that $fail(x,t) < \frac{1}{2^n}$ and relation (3) we get that: $\frac{U}{V} < \frac{|kw_{x,t}|}{|k_{x,t}|} < \frac{1}{2^n}$.

Therefore, $|kg_{x,t}| > \frac{9 \cdot 2^n \cdot U}{10} - U = U(\frac{9 \cdot 2^n}{10} - 1)$.From $eU < m < 2eU$, we finally reach: $|kg_{x,t}| > \frac{m}{2e} \cdot \left(\frac{9 \cdot 2^n}{10} - 1\right) > pm$, which is exactly what we wanted to show. □

We now discuss the implications of THEOREM 10.3.

OBS 7: Consider any *reliable* transcript t selected from S_n and C a (h,p)-*collision set* as constructed above. THEOREM 10.3 tells us there are 2 possibilities:

i) either the set of *good* keys corresponding to t contains the subset C of p keys all mapped by h to the same value or
ii) C contains both *good* keys and *wrong* keys (because the second case of THEOREM 10.3 tells us there can be no (h,p)-*collision set* containing *only wrong* keys).

Therefore, for any (h,p)-*collision set* C we can verify if $C \subseteq k(x,t)$. If the *outcomes* obtained for $f(x)$ from all keys in C are *equal* (meaning that all keys in C are *good*), then we can conclude that we are in case i) and we know for sure that this *outcome is the real* $f(x)$.
If we were in case ii), because C had both *good* and *wrong* keys, we would have at least 2 different outcomes for $f(x)$ resulting from 1 *good* key and 1 *wrong* key.

Algorithm to solve *f*

Finally, we are able to define the $NP/poly$ algorithm which computes $f(x)$.

As suggested before, the advice received for all inputs of length n consists of the set of transcripts $S_n = \{t_1, \ldots, t_w\}$ and for each transcript t_i the advice also contains the associated universal hash function (h_i, p_i, m_i).

The ***NP* algorithm** can be described as follows:

- We receive the advice S_n containing a majority of *reliable* transcripts.
- The nondeterministic step tries every subset of S_n of size larger than $\frac{|S_n|}{2}$ until we find the one consisting of the *reliable* transcripts of S_n.
- Now, for each transcript t_i from this subset we must find a $(h_i,\, p_i)$-collision set $C = \{k_{i_1}, k_{i_2}, \ldots, k_{i_{p_i}}\}$ containing only *good* keys. We use another **nondeterministic choice** to identify C. To check that C is a $(h_i,\, p_i)$-collision set, we look at the mappings $h_i(k_{i_j})$ and we test if they are all equal. Then, as described in OBS 7, we know that C only consists of *good* keys if the outcomes obtained for $f(x)$ are all equal to the same value, call it y_i.
- Finally, we verify that for all transcripts t_i we have the **same outcome $\boldsymbol{y_i}$**. We conclude that this value is exactly the **desired $\boldsymbol{f(x)}$**.

This algorithm ends the proof for the MAIN THEOREM 1 [16] by showing that f can be computed by an $NP/poly$ algorithm. □

We have seen which are the implications for solving any problem f using the *GES* framework. Our main target is to decide whether a function $f \in BQP$ could be computed in this scenario. But before that, we investigate a similar protocol called *Universal Blind Quantum Computing*, which has similar aim: to securely delegate a quantum computation. One of the main reasons why we consider this protocol is because *Universal Blind Quantum Computing* can solve any BQP problem [15].

Consequently, we will investigate and in fact develop an adaptation of the *GES* scheme in order to match the *Universal Blind Quantum Computation* definition.

5. Universal Blind Quantum Computing

Universal Blind Quantum Computing (*UBQC*) is a quantum protocol proposed by Broadbent, Fitzsimons and Kashefi [26], in which a client with limited computational resources asks a server owning a quantum computer to solve a problem for him. The client gets the result of his computation in such a way that the server cannot infer anything about the input or even about the computation required by the client. In this scheme, the client must be able to prepare single qubits and then send them to the server, but does not need any other quantum resources.

UBQC is an interactive protocol where the client and server exchange messages [11, 12]. In order to obtain the desired computation outcome, the client must indicate to the server what qubit measurements to perform depending on past measurements results. Additionally, the scheme also allows the client to address a problem which has either classical or quantum inputs/outputs [15].

UBQC protocol is constructed using the *Measurement Based Quantum Computing* model [29]. It creates a generic pattern for all quantum computations and sets apart the quantum and classical components of a given problem. As a result, the client does not need to make any quantum computation or to own quantum memory.

Furthermore, we can also identify a malicious server which might try to deceive the client. Keeping the input and the computation private from the server is not based on the difficulty of solving hard problems (as it is the case for current classical cryptographic systems [9]) and its security can be maintained irrespective of what a malicious server might try to do. If the client would require the solution for an NP problem, then he could check if the result received from the server is correct (solutions to NP problems can be verified in polynomial time [17]). In this way, he would find out if the server should be trusted or not. However, for harder problems and in the cases where the server interferes with the computation [12], the client may be deceived.

This scheme is *universal*, meaning that it can work for any quantum computation [6]. For instance, suppose the client would like to use the computational power of the server for the problem of *factorization*. Then, using the Shor's algorithm [7] in the form of a blind computation, will allow the client to obtain the factorization, without the server knowing the input or even the fact that he is factorizing a number. In fact, this scheme can be used for any $BQP - complete$ problem, such as *k-Forrelation* [25](which will be discussed later).

UBQC can be understood as an *interactive Measurement Based Quantum Computing* [26], where both the client and the server contribute, in order to perform the computation:

1) The client needs to build qubits
2) The server generates the entangled resource and completes the measurements
3) The client is instructing the server upon the next measurements and is also adding some randomness to keep the computation private from the server.

Now, suppose in the *MBQC* model the client would use the highly entangled graph state [1, 29]. As we remember, the graph state consisted of multiple qubits from which we derived our computation by making measurements in either the $\{|0\rangle, |1\rangle\}$ or in the $\{|+_\theta\rangle, |-_\theta\rangle\}$ basis [30]. The problem with this approach is that we might need different graphs for different problems, hence the client would leak some information about the desired computation.

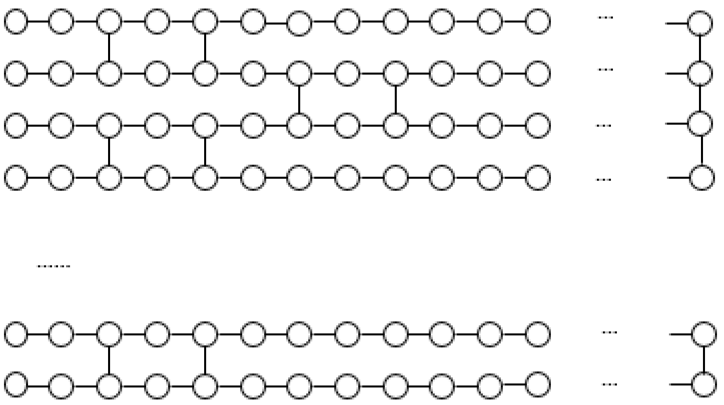


Figure 3. Brickwork state, where each cell corresponds to a qubit in the $|+\rangle$ state

We remind from the *MBQC* model, that measurements performed in the $\{|0\rangle, |1\rangle\}$ basis affect the pattern of the graph state, in the sense that they remove qubits from the graph in order to match the design pattern required for the computation [29]. On the other hand, any measurement performed in the $\{|+_\theta\rangle, |-_\theta\rangle\}$ basis was used to implement the computation and would leave the graph unchanged.

Instead of working on a graph state, we use as starting point *a brickwork state* (Fig. 3) [26]. This state has the advantage that by performing only measurements in the $\{|+_\theta\rangle, |-_\theta\rangle\}$ basis we can obtain any quantum operator. Even more, this state has the same pattern for all quantum computations and only depends on the size of input and size of computation. In the *MBQC* scenario, the measurements destroyed the initial graph state (removing certain qubits from the graph), so a malicious server can deduce the pattern of the graph. On the other hand, in the situation of using the *brickwork state* we only need to measure in the $\{|+_\theta\rangle, |-_\theta\rangle\}$ basis, so the brickwork state is not altered, remaining generic for all computations. Thus, it does not leak any information about the problem [15, 27].

Now, we will present how does the *Universal Blind Quantum Computing* work and what are the properties of this protocol.

Suppose the client A has a problem $\mathcal{P}$, which he wants to solve using *UBQC*. Then, A needs to have a description of the circuit using the brickwork state G (Figure 3). Client A randomly constructs qubits in the states $\{\frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle)|\, \phi \in \left\{0, \frac{\pi}{4}, \dots, \frac{7\pi}{4}\right\}\}$ and delivers them to the server B. Once B receives the qubits, he must entangle them in accordance with the state G.

The brickwork state can be viewed as a matrix of qubits. For each line and for each qubit of the line, A will tell B which is the angle of the measurement for that particular qubit [27]. As a result, B completes the measurement and announces A about the result. Then, A will do the same thing for the next qubits, but their angles determined by A, are based on the previous results received from B.

A obtains the desired computation result once all the qubits in the brickwork state are measured [15].

Consider that for the problem $\mathcal{P}$ we have the brickwork state G having l lines and c columns. Then, each qubit $|\phi_{i,j}\rangle$ of this state is initially in the $|+\rangle$ state and it has associated a measurement angle $\rho_{i,j}$. As in the *MBQC* model, these angles are affected by previous measurements: any $\{|+_\theta\rangle, |-_\theta\rangle\}$ measurement implies possible X corrections changing an initial angle ρ to –ρ and Z corrections changing an initial angle ρ to ρ + π [15]. Therefore, we have that the actual angles for every qubit are: $\rho'_{i,j} = (-1)^s \rho_{i,j} + t\pi$, where s is the number of previous X corrections and t is the number of previous Z corrections [29].

We can view the problem $\mathcal{P}$ as a quantum operator U and so, the client A wants to obtain the result $U|0\rangle$. The *UBQC* scheme for computing U is the following [26]:

- *Stage 1: Qubits Setting*

  For every column c of G and for every line l of G:

  Client A sets the qubit $|\phi_{c,r}\rangle$ randomly chosen from the set

  $S = \{\frac{1}{\sqrt{2}}(|0\rangle + e^{i\theta_{i,j}}|1\rangle)|\, \theta_{i,j} \in \left\{0, \frac{\pi}{4}, \dots, \frac{7\pi}{4}\right\}\}$ and sends $|\phi_{c,r}\rangle$ to the server B;

  Using the qubits obtained from the client, the server builds the entangled state. This is achieved by applying the operator *Controlled-Z* between pairs of qubits, thus defining the structure of G.

- *Stage 2: Qubits measurement and communication client-server*

  For every column c and line l of G:

  The client computes the standard angle of measurement $\rho'_{i,j}$ in order to obtain U

  A also adds an element of randomness to hide the computation from B, $rand \in \{0, 1\}$ and sets the angle of measurement as: $\delta_{i,j} = \rho'_{i,j} + \theta_{i,j} + rand \cdot \pi$.

  This new angle is received by the server B. Consequently, B measures the qubit in the basis $\{\frac{1}{\sqrt{2}}(|0\rangle + e^{i\delta_{i,j}}|1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - e^{i\delta_{i,j}}|1\rangle)\}$ and obtains the result $r \in \{0, 1\}$ which is sent to the client.

  The client needs to modify the result r to compensate for the randomness he added, so the result of the measurement becomes $r' = |rand - r|$

The *Universal Blind Quantum Computing* protocol has 3 important properties: **universality**, **correctness** and **blindness** [6, 9], which we will next define.

DEFINITION 6.1: *Universality* refers to the fact that the scheme allows the computation of any quantum operator [6]. This is achieved due to the fact that the *MBQC* model is *universal* and in fact we can obtain this property by only performing measurements with the angles: $\left\{-\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}\right\}$. ☐

DEFINITION 6.2: Showing that the result returned by the protocol is indeed $U|0\rangle$ means the protocol is *correct* [15].

The stage 1 of the protocol involves the rotations of the qubits around the Z-axis with the angle θ and the application of the Controlled-Z gate on pairs of qubits. These operations do not affect the brickwork state (as opposed to the graph state case), but only modify the phase of qubits as if the *Controlled-Z* gate was first applied and then the qubits were rotated with the angle θ.

Furthermore, A changing the angle of measurement from ρ' to $\rho' + \theta$ is equivalent to adjusting the state on which we applied the measurement from $|\phi\rangle$ to $Z(\theta)|\phi\rangle$. Therefore, as A's choice for the new measurement angle is defined as: $\delta_{i,j} = \rho'_{i,j} + \theta_{i,j}$ + *rand* · π, then if $rand = 0$ the server's outcome r is exactly the result A should have and if $rand = 1$, then A just needs to change the result to $1 - r$. ☐

DEFINITION 6.3: The *blindness* property refers to the fact that B cannot infer anything about A's input or the actual computation (except their size), no matter what B's actions might be [26].

We can view this task of blind computation in the following perspective: the client A needs to solve the problem $\mathcal{P}$ by exchanging messages with the quantum server B. Then, for a given input x and the description of $\mathcal{P}$ as an operator U, A's input for the protocol becomes $I = (U, x)$.

DEFINITION 6.4: We say a delegated quantum computation acting on input I is *blind hiding everything except $H(I)$* [27] if the knowledge inferred by a malicious server B has no connection with the input I and the quantum state of B is also uncorrelated with I.

THEOREM 11. We are now able to show that the *UBQC* scheme is a *delegated quantum computation hiding everything but the size of the brickwork state* [15](the number of lines l and the number of columns c).

Proof. As we have already shown, the generic brickwork state allows the computation of any operator U, without being modified in any way by the measurements. Therefore, we do not reveal any information about U, except the size of the problem, l and c.

The client A has the initial measurement angles $\rho_{i,j}$ for every qubit, needed to obtain U. Then, these angles are altered based on the outcome of the previous measurements executed by B, to form the new angles: $\rho'_{i,j} = (-1)^s \rho_{i,j} + t\pi$. A does not send to B exactly these angles, but instead, the classical knowledge B obtains is represented by the measurement angles: $\delta_{i,j} = \rho'_{i,j}$ + $\theta_{i,j}$ + *rand* · π.

Therefore, what we need to prove is that $\delta_{i,j}$ (the classical knowledge of B) is not connected with $\rho_{i,j}$ (the input of the problem).

We know the fact that $\theta_{i,j}$ is randomly drawn from the set $\left\{0, \frac{\pi}{4}, \dots, \frac{7\pi}{4}\right\}$. Thus, because $\delta_{i,j} = \rho'_{i,j} + \theta_{i,j} + rand \cdot \pi$, we have that $\delta_{i,j}$ cannot be based on $\rho_{i,j}$ [29].

At this point, we analyse the quantum state of B resulted after A's qubits preparation, call it S.

For each qubit of S, $|\phi_{i,j}\rangle$ we need to consider 2 cases:

- If $rand = 0$, then the angle received by B is $\delta_{i,j} = \rho'_{i,j} + \theta_{i,j}$ and thus, the qubit state is: $|\phi_{i,j}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\theta_{i,j}}|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + e^{i(\delta_{i,j}-\rho'_{i,j})}|1\rangle)$
- If $rand = 1$ then the angle received by B is $\delta_{i,j} = \rho'_{i,j} + \theta_{i,j} + \pi$ and thus, the qubit state is: $|\phi_{i,j}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i(\theta_{i,j}+\pi)}|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle - e^{i(\delta_{i,j}-\rho'_{i,j})}|1\rangle)$

We observe that if we fix the value of $\delta_{i,j}$, then $\theta_{i,j}$ will depend on $\rho'_{i,j}$ ($\theta_{i,j} = \delta_{i,j} - \theta_{i,j} + rand \cdot \pi$). However, $rand$ is chosen independently at random from $\{0, 1\}$, therefore S contains qubits from the brickwork state, which are not based on the input $\rho_{i,j}$. $\square$

We can now illustrate the usefulness of this *blindness* property of *UBQC* by presenting 2 attack examples [26, 27].

Example 1: Assume B owns some additional prior information about the input I. But using the blindness definition, it results that this prior information is not connected with I and therefore, the only knowledge inferred by B is the leaked information, the size of the computation.

Example 2: Assume B aims to find some information about the output of the computation. However, because the output would be based on the input I and we know that B cannot find out anything about the input, then this situation is also impossible.

6. Classical Universal Blind Quantum Computing

In the *Universal Blind Quantum Computing* protocol, the client does not need to possess quantum memory or the ability to perform quantum computations, however, he must be able to prepare qubits in different states, so he is not an *entirely classical* client [26, 29].

What we want to investigate is the possibility of the *existence* of a "*Classical Universal Blind Quantum Computing*" (*CUBQC*) scheme, where the client is *purely classical* and where the *interaction* between the client and the quantum computer must also be *classical*.

A similar scheme for this type of *blind delegated computation* involving one entirely classical client was proposed by Broadbent et al [15]. Unfortunately, this protocol requires 2 entangled quantum servers: the first one completes the quantum work done by the client in the *UBQC* scenario (prepares randomly selected qubits) while the second one performs the actual computation.

This classical *UBQC* protocol which we propose must obey the *UBQC* scheme definition according to whom the *input*, *output* and the *computation* itself must be *private* to the quantum server. Moreover, it should respect the 3 main properties discussed before: *universality, correctness and blindness* [27].

We relate the existence of *CUBQC* with the *Generalised Encryption Scheme*. To be more specific, we will prove that *CUBQC* can be represented as an instance of the *GES* framework and thus, this would imply that any major result known for *GES* is also applicable to *CUBQC*.

To begin with, by comparing *CUBQC* with *GES*, we notice that the major difference between the two interactive delegated computation schemes is the following:

In *GES,* the server already knows the problem needed by the client and we use the encryption procedure to hide only the input and the output of a problem, whereas in *UBQC* the actual computation is also kept secret from the server.

Therefore, we need to make the modifications such that in the *GES* scenario we also **encode** the actual computation in the input string given to the encryption algorithm, with the resulting cyphertext being sent afterwards to the server.

At this moment is worth mentioning a brief review of the main *GES* principles [16]:

$$\text{Client } A \xrightarrow{y=E(x)} \text{Server } B$$

$$\text{Client } A \xleftarrow{f(y)} \text{Server } B$$

The client A needs to determine the value of a function f applied on an input x. He encrypts this input, $y = E(x)$, and sends the encryption y to the server. The server then applies the function f on y and sends the result back to A. Finally, A only needs to apply the decryption function to obtain $f(x)$. Therefore, the encryption and decryption primitives must satisfy the important property that: $D\left(f\big(E(x)\big)\right) = f(x)$.

Furthermore, because the client A has only limited computational resources the encryption and decryption functions must run in *polynomial time*.

Before proceeding to the main proof, we need to outline some **essential notions** which will be used in our arguments.

DEFINITION 7. A *Quantum Turing Machine* (QTM) is the correspondent in the quantum perspective of the Probabilistic TM [6]. The classical probabilistic machine evolves according to a probability distribution describing the likelihood of every future state of the machine. In the quantum machine case, we are dealing with a *quantum superposition* of possible states where every coefficient of a state represents its *amplitude* [10].

The internal states of a QTM M are represented by states in the *Hilbert Space* [1], while the transition function is a unitary operator. M can be described by the tuple $(\Sigma, Q, \delta, q_0, q_f)$, where Σ is the set of input symbols, Q is M's set of states from the *Hilbert Space*, δ is the *transition function* of the machine, q_0 is the *initial state* and q_f is the *accepting state*.

M halts on an input x if the machine state gets in a superposition of possible configurations, all in the accepting state q_f. We say that M is a *polynomial-time Quantum* TM if it halts on every possible input x and the running time of the machine is polynomial in $|x|$ [10, 17].

Using these descriptions, we are now ready to define a ***Universal Quantum Turing Machine*** (the quantum version of the classical Universal TM).

A *Universal Quantum Turing Machine* would allow us to *simulate* any QTM M, by describing the specification of M and sending it as an input to the universal machine.

Bernstein and Vazirani proved the existence of this generic machine [10], we will call it M_U. Specifically, M_U receives as input: the representation of a QTM M, an input x for M and a precision $\mathcal{E}$ and returns a quantum superposition ψ, such that the distance between ψ and the superposition of M applied on x is less than $\mathcal{E}$.

The way M_U functions is very straightforward: it is based on a deterministic Turing Machine extended with only one quantum procedure: a quantum analogue of a coin flip which rotates individual qubits [1]. It was proved that in order to simulate any quantum TM it suffices to apply this operation using the amplitudes $\frac{3}{5}$ and $\frac{4}{5}$ [30]. Additionally, the running time of M_U is polynomial in the length of the input.

What is left, is to define the representation of any QTM as a string which would then be given as part of the input to M_U. For a given QTM M we can encode it in a string S which contains the following: the finite alphabet Σ, the number of states from Q and the description of the transition function δ.

Every input of the δ function consists of a tuple (q_1, s_1, s_2, q_2, d), where q_1 represents the current state of M, s_1 is the symbol on the infinite tape, indicated by the tape's head, s_2 is the symbol M will write on the tape, d expresses whether the head of the tape is moving left, right or remains at the same position and q_2 is the next state of M. The transition can be illustrated like this: suppose M is in state q_i. Then, at next step the machine will be in the quantum superposition ψ = $\sum_j q_j \cdot \alpha_j$, with q_j representing a possible future state and α_j the amplitude associated with a transition tuple $(q_i, s_{i,1}, s_{i,2}, q_j, d)$ (${\alpha_j}^2$ is the probability of this transition) [20, 30]. Therefore, the encoding string S must contain the amplitudes associated with all these possible tuples (which are in number of $3 \cdot |Q|^2 \cdot |\Sigma|^2$).

Now we can return to our problem and describe one of our **primary contributions**.

THEOREM 12. *CUBQC can be represented as an instance of the GES protocol.*

Proof. In the *GES* scheme the server holds the function f and he applies it on whatever input (encrypted input) he receives from the client.

We change f so that instead of denoting the computation required by the client, f will represent a Universal Quantum Turing Machine M_U.

Suppose the client wants to compute a problem $\mathcal{P}$ for the input x. We know that there must exist a $QTM\ M$ which is able to solve $\mathcal{P}$. Then, we can encode the behaviour of M in a string S, using the method described above. It is worth mentioning that the encoding can be done efficiently in polynomial time, so the client having only BPP computational power can perform this procedure.

Then, the client's input I would consist of S and the input x for M, I = (S, x). From this point, the client proceeds as in the *GES* scheme. He applies the encryption primitive E to obtain $J = E(I)$ and sends J to the server.

The server uses J as input for the generic $UQTM\ M_U$ and sends back to the client the outcome returned by M_U. Finally, the client applies the decryption function on the outcome received from the server and obtains the desired result $f(x)$.

The encryption and decryption procedures must satisfy the property: $D(M_U(E(S, x))) = f(x)$. $(*)$

In this way, by using the *Universal* QTM we assured that the server cannot find anything about the computation needed by the client, since the problem is also sent to the server as an encrypted input.

We need *to verify* if the 3 properties assured in the *UBQC* protocol are also available here.

The correctness property, which refers to the fact that the client obtains in the end $f(x)$, is guaranteed by the *GES* scheme, specifically by the decryption and encryption functions property $(*)$.

Universality is obtained because the $UQTM\ M_U$ allows the simulation of any $QTM\ M$, therefore the client can use this scheme for any computation [10].

Because the *GES* protocol has the privacy property that the server cannot infer anything about his input except its length [16], this implies that *CUBQC* is blind, leaking only the size of the computation and the length of the input, which is exactly the level of security guaranteed in the *UBQC* protocol.

Consequently, we defined the *Classical UBQC* scheme, represented as an instance of *GES.* ☐

Therefore, any *complexity theoretic result* regarding classes of problems which can or cannot be solved using *GES* also *applies* to the *CUBQC* protocol.

Let's consider that the client A wants to compute a problem $\mathcal{P}$ on an input x, where $\mathcal{P}$ is from the BQP class. Because $\mathcal{P} \in BQP$, there exists a *polynomial-time* $QTM\ M$ which can solve $\mathcal{P}$ [19]. Then, A encodes the representation of M and obtains the string $S = \rho(M)$. A encrypts S along with the input x and sends the resulting encryption e to the server B. B runs the $UQTM\ M_U$ on the input e and sends back to A the outcome of M_U. In the end, A decrypts this outcome and gets $\mathcal{P}(x)$. However, we will next present strong evidence that the *GES* scheme cannot solve $BQP - hard$ problems. This tells us that the **no-go result** also **stands for *Classical UBQC***, namely that there does **not exist any *CUBQC*** scheme which could **solve for $\boldsymbol{BQP - hard}$** problems.

Our Main Results

Next, we will proceed to the central part of the research project which depicts our most important contribution: analyse the possibility of $BQP - hard$ problems being solved using *GES*.

We base our proofs on the main result of Abadi obtained for *GES* (described in MAIN THEOREM 1), according to which any problem solvable using GES must belong to the class $NP/poly \cap coNP/poly$. Then, the statement "No $BQP - hard$ problems can be solved using *GES*" is equivalent to showing that $BQP - Hard \not\subseteq NP/poly \cap coNP/poly$. Our target is to give strong evidence for this *major complexity-theory result*.

In the following 2 sections, we explore two problems which indicate the **oracle separation** between the classes BQP and $NP/poly \cap coNP/poly$, namely: *Simon's* problem and *k-Forrelation*.

7. Simon Problem

Simon problem is a decision problem involving a "*black-box*" [21] function f, where we need to determine what kind of function is f given a promise about it. More exactly, we are promised that f is either *1-to-1* or *2-to-1* and we need to determine whether f is of the former or latter type [23].

DEFINITION 8.1: We call a function $f: \{0,1\}^n \to \{0,1\}^n$ *1-to-1* if f is bijective.
DEFINITION 8.2: We say a function $f: \{0,1\}^n \to \{0,1\}^n$ is *2-to-1* if it satisfies the following property:

$\exists s \in \{0,1\}^n$- $\{0\}$ such that $\forall x, x'$ we have: $f(x) = f(x') \Leftrightarrow x' = x \oplus s$. We call s the *xor mask* of f.

f is a "*black-box*" function refers to the fact that f is given in the form of an *oracle* (we cannot see its explicit definition, but we can ask the oracle for values of f at different points).

Classical Algorithm for Simon

In order to solve this problem deterministically, we would need $O(2^n)$ queries [23]. For instance, we could search for an element m, $1 \le m \le 2^n - 1$ such that $f(0) = f(m)$. If we find such an m, then f must be *2-to-1* (it cannot be bijective) and its *xor mask* is m. Otherwise, if no such m exists we conclude that f must be bijective. However, using a classical randomized algorithm we can obtain a better time-complexity [28], solving this problem with only $O(\sqrt{2^n})$ queries. We pick some input values x_i at random, and for each, we consult the oracle to get $f(x_i)$. Once we find a *collision pair*, namely 2 input values x_i and x_j such that $f(x_i) = f(x_j)$, we know f is *2-to-1* and $s = x_i \oplus x_j$. Using the "birthday paradox"[4] [34] we achieve that with only $\sqrt{2^n}$ queries the problem can be solved. Moreover, it can also be proven that since f can only be accessed through the oracle which computes its values, the classical query complexity $\sqrt{2^n}$ is optimal.

[4] The probability that 2 persons have the same birthday. Given there are 365 days in a year and birthdays are almost uniformly distributed, what matters is the number of pairs of people: if we have $\sqrt{n}$ persons, where n is the number of days in a year, there are n different pairs of people who can have the same birthday and with high probability one of them will succeed.

7.1. Quantum Algorithm

In the quantum paradigm, this problem can be solved using n queries to the oracle O, thus showing an exponential speedup compared to its classical probabilistic counterpart [30].

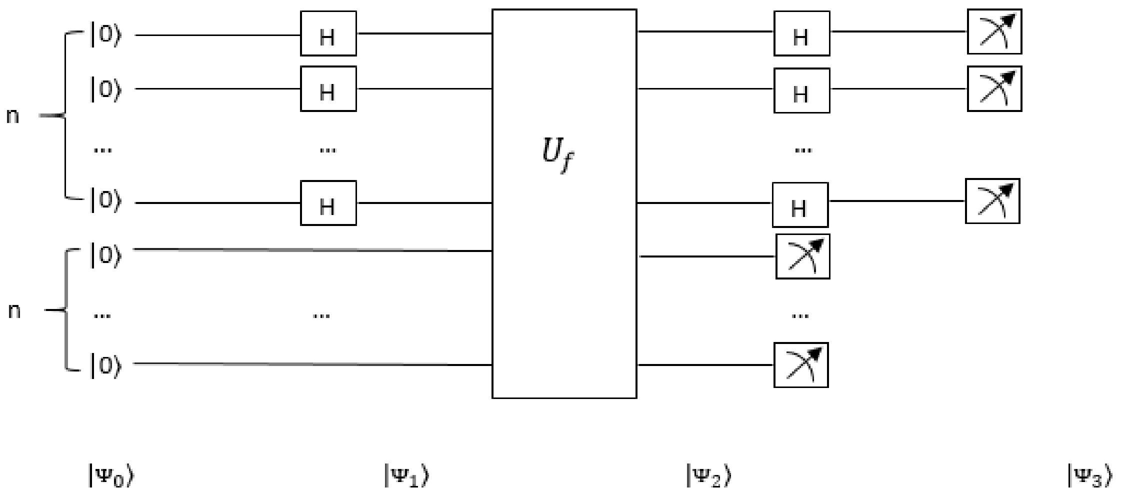


Figure 4. Quantum Circuit for Simon's Algorithm

We can prove our problem is in the BQP class by presenting an expected quantum polynomial-time algorithm which always gives the correct answer for *Simon* problem.

MAIN THEOREM 2.1. $Simon\ Problem \in BQP^{O}$

Proof. In order to solve this problem, we are going to use *Simon's* algorithm [23] (Figure 4).

The initial state is $|\Psi_0\rangle = (|0\rangle \otimes \dots \otimes |0\rangle) \otimes (|0\rangle \otimes \dots \otimes |0\rangle) = |0\rangle^{\otimes n} \oplus |0\rangle^{\otimes n}$ (2 registers of n qubits)

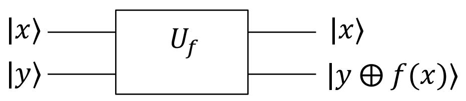
The first thing we do is create an equal superposition over all 2^n possible inputs to f and we achieve this by applying n *Hadamard* operators on the first register.

After applying the n *Hadamard* gates on the first n qubits (first register), we get the state:

$$|\Psi_1\rangle = |+\rangle^{\otimes n}\,|0\rangle^{\otimes n} = \frac{1}{\sqrt{2}^n}(|0\rangle + |1\rangle)^n\,|0\rangle^{\otimes n} = \frac{1}{\sqrt{2}^n}\left(\sum_{x \in \{0,1\}^n} |x\rangle\right) \otimes |0\rangle^{\otimes n}$$

The next step is querying f.

Applying the *black-box* operator for f has the following effect [29]:



Thus, in our case we obtain:

$|\Psi_2\rangle = \frac{1}{\sqrt{2}^n} \sum_{x \in \{0,1\}^n} |x\rangle \otimes |0^n \oplus f(x)\rangle = \frac{1}{\sqrt{2}^n} \sum_{x \in \{0,1\}^n} |x\rangle \otimes |f(x)\rangle$ - we have reached a superposition of all values of f in the second register.

As we can see in the circuit, in state $|\Psi_2\rangle$ we will measure the second register which contains $f(x)$.

After the measurement, what is left in the first register, will be a superposition over all possible inputs x that could have produced the $f(x)$ we observed. This superposition is $\frac{1}{\sqrt{2}}(|x\rangle + |y\rangle)$ where x and y are 2 inputs with the property that $f(x) = f(y)$.

Next, we apply n Hadamard gates on the first register (which now only consists of $\frac{1}{\sqrt{2}}(|x\rangle + |y\rangle)$). Applying Hadamard on an input $|x\rangle$ consisting of n qubits has the following effect [1]:

$|x\rangle \xrightarrow{H^n} \frac{1}{\sqrt{2}^n}\sum_{w \in \{0,1\}^n}(-1)^{x \cdot w}\,|w\rangle$, where if we have $x = x_1 \dots x_n$ and $w = w_1 \dots w_n$,

then $x \cdot w = x_1 w_1 \oplus x_2 w_2 \oplus \dots \oplus x_n w_n$.

When we apply Hadamard to $\frac{1}{\sqrt{2}}(|x\rangle + |y\rangle)$ we get:

$$\frac{1}{\sqrt{2}}(|x\rangle + |y\rangle) \xrightarrow{H^n} \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}^n}\sum_{w \in \{0,1\}^n}(-1)^{x \cdot w}\,|w\rangle + \frac{1}{\sqrt{2}^n}\sum_{w \in \{0,1\}^n}(-1)^{y \cdot w}\,|w\rangle\right) =$$

$$\frac{1}{\sqrt{2}^{n+1}}\sum_{w \in \{0,1\}^n}[(-1)^{x \cdot w} + (-1)^{y \cdot w}]\,|w\rangle$$

Therefore, we obtained a superposition of all inputs w, each with amplitude $(-1)^{x \cdot w} + (-1)^{y \cdot w}$.

Finally, we measure each of these qubits from the first register in the $(|0\rangle, |1\rangle)$ basis [12].

We notice that if $(-1)^{x \cdot w}$ and $(-1)^{y \cdot w}$ are equal, then it is possible to observe the corresponding state w after the measurement.

But, if $(-1)^{x \cdot w}$ and $(-1)^{y \cdot w}$ are different, they cancel each other, so w cannot be observed.

Hence, after the measurement we are going to be left with a random input w, such that: $(-1)^{x \cdot w} = (-1)^{y \cdot w}$. We can rewrite this relation as:

$(-1)^{x \cdot w} = (-1)^{y \cdot w} \Leftrightarrow x \cdot w \equiv y \cdot w \; mod\; 2$,

which is equivalent to $(x \oplus y) \cdot w \equiv 0 \; mod\; 2$.

But $x \oplus y = s$, thus we obtain $s \cdot w \equiv 0 \; mod\; 2$.

Consequently, we have a random linear equation satisfied by the *xor mask* s.

We can repeat this whole process (run again *Simon's* algorithm) to obtain a second linear equation:

$$\begin{cases} s \cdot w_1 \equiv 0 \; mod\; 2 \\ s \cdot w_2 \equiv 0 \; mod\; 2 \end{cases}$$

In this way, if we repeat the algorithm $O(n)$ times, we will get with high probability n linear equations, independent of each other and we can efficiently solve this system of linear equations [30] (for instance using Gaussian Elimination).

Then, if the system has only 1 solution $s = 0$, this means that f must be bijective. Otherwise, we will obtain 2 solutions, $s_1 = 0$ and $s_2 \neq 0$, where s_2 represents the *xor mask* for f.

Every time we run *Simon's* algorithm we query f only 1 time, so in total we need $O(n)$ queries to f.

As a result, we have proved that *Simon's* problem can be solved by a quantum computer in polynomial time with bounded error probability, using an oracle O which returns the values of f:

$Simon\ Problem \in BQP^O$ □

7.2. Classical no-go Result

We remind that our goal is to show that $BQP \nsubseteq NP/poly \cap coNP/poly$. We want to give strong evidence in favour of this relation by proving that it is true with respect to an oracle O:

$$BQP^O \nsubseteq NP/poly^O \cap coNP/poly^O$$

This is the reason why we are looking for a problem which can be shown to be solvable in BQP^O, but which cannot be solved in $NP/poly^O \cap coNP/poly^O$.

The BQP class is *closed under complement* [17], which means that the complement of any problem from BQP is also in BQP. Therefore, the **complement of *Simon's* problem** (*coSimon*) belongs to the BQP class. We can obtain *coSimon* by *reversing* the *yes* and *no* answers to *Simon's* problem [20]:

The *coSimon* decision problem will return "*yes*" if a function f is a bijection and "*no*" if f is a *2-to-1* function.

From this point on, we will be working only with the ***coSimon*** problem. Our aim is to prove that *coSimon* is not in the set $NP/poly^O$, hence it cannot be in the set $NP/poly^O \cap coNP/poly^O$ either.

The reason why we are considering *coSimon* and not directly the *Simon* problem is the following:

Simon problem can be solved by a TM in NP^O, so it also belongs to the $NP/poly^O$ class. Therefore, we would have needed to prove that it is not in $coNP/poly^O$. But, while the complement of NP is the class $coNP$, $(coNP)^O$ is **not** the complement of the class NP^O. Since we do not have a definition of the class of problems represented by $(coNP)^O$ or by $(coNP/poly)^O$, we will instead work with the complement of *Simon's* problem and show that it cannot be in $NP/poly^O$.

7.2.1. NP-Impossibility

We begin by demonstrating the *coSimon* problem is not in the complexity class NP^O.

MAIN THEOREM 2.2: $Simon\ Problem\ \notin NP^O$.

Proof. Assume *coSimon* $\in NP^O$. Then, there exist an *Oracle Turing Machine* $M \in NP$ which accepts a function f when f is bijective and rejects f when f is *2-to-1*.

As this is an oracle problem, the input for M will not be an explicit description of a function f [21, 22] (for example, the value of f in every element of his domain).

Instead, we use the fact that the set of *1-to-1* and *2-to-1* functions $f : \{0,1\}^n \to \{0,1\}^n$ is **countable** and thus we can associate for any such function a natural number **index**.

Then, M will receive as input the pair (i, n), where i represents the index of the function and n (which is written in the *unary* system) specifies the size of any element from the domain of the function: (i, n) maps to $f_i : \{0,1\}^n \rightarrow \{0,1\}^n$.

The oracle O works in the following way: it receives the index i of the function, the length n and an element $x \in \{0,1\}^n$ and returns to M the value of $f_i(x)$ in only one computational step: $O(n, i, x) = f_i(x)$.

Suppose we would have a random oracle O [22]. Then, given the length n, O is either uniformly distributed among bijective functions or is distributed uniformly among *2-to-1* functions.
For each possible length n we have an oracle O_n and suppose we have $\frac{N_n}{2}$ bijections and $\frac{N_n}{2}$ *2-to-1* functions chosen for O_n, with $N_n = O(2^n)$.
Then we can consider our oracle O as the union of all oracles O_n $\forall n \in \mathbb{N}$. O has $\frac{N}{2}$ bijections and $\frac{N}{2}$ *2-to-1* functions ($N = \sum_i N_i$).
Consequently, given an index i we have $P(f_i$ is bijective) = $P(f_i$ is *2-to-1*) = $\frac{1}{2}$.
For instance, the oracle O could be built in the following way: $\forall n \in \mathbb{N}$ and for $i \in \{1, \dots, 2 \cdot N_n\}$ we randomly selected a bit $b_{n,i}$ and a string $s_{n,i}$ of length n. If $b_{n,i} = 1$ then the function $f_i : \{0,1\}^n \rightarrow \{0,1\}^n$ chosen to be computed by oracle O, is 2-to-1 with the *xor mask* $s_{n,i}$.
If $b_{n,i} = 1$ then the function $f_i : \{0,1\}^n \rightarrow \{0,1\}^n$ is bijective.

Instead of having a random oracle, the oracle used for our problem is **adversarial** [31]. This implies that O does not hold any N 1-to-1 or 2-to-1 functions chosen uniformly at random, but O is **allowed to choose** N_n 1-to-1 or 2-to-1 functions for any length n (out of the $(2^n)!$ total number of permutations and out of the $(2^n - 1)^{2^n} \cdot 2^n$ total number of *2-to-1* functions).

Consequently, we need to **build** the oracle O (select the N_n functions owned by O) in such a way that $coSimon^O \notin NP^O$.

Our proof is based on **diagonalization** [16].

The set of nondeterministic TM is countable [21] so we can consider an enumeration of them: $M_1, M_2, \dots, M_k, \dots$.

Therefore, we must construct the adversarial oracle O such that:

$\forall i \in \mathbb{N}$ $coSimon^O \notin M_i{}^O$

Initially,the set of functions chosen for O, $\mathcal{F}$ is empty and at each step i we add another function to it.

Suppose we are at step i and we have the oracle nondeterministic TM $M_i{}^O$.

The adversarial oracle O knows the computational tree of M_i. We can view M_i as an algorithm which uses the oracle to take decisions for the paths in its computational tree. The root of the tree contains the initial configuration of M_i on an input, whereas all the internal nodes are queries made to the oracle. When no more queries are made on a path, we reach a final node (a leaf), which is either an accepting or a rejecting state [20].

The input for M_i consists of the *index* of the function and the *size* of any element from the function's domain. Thus, for the input $(index, n)$, the computational tree of M_i would look as seen in Figure 5:

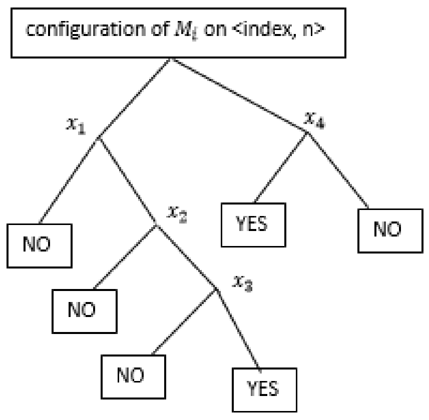


Figure 5. Computational tree of M_i on input <index, n>

We now describe O's behaviour which must make sure that M_i^O cannot always solve correctly *coSimon.*

The oracle picks any *1-to-1* function $g : \{0,1\}^n \to \{0,1\}^n$, which is correctly accepted by M_i. This means that if the input to M_i is $(index, n)$, the oracle will place g on the position $index$ inside his function set $\mathcal{F}$. M_i accepts g is equivalent to: there exists a path in the computational tree of M_i on input $(index, n)$ which ends in an accepting state. This path contains l strings $x_1, \dots, x_l$ for which M_i queries O. We call this path $\pi = x_1, x_2, \dots, x_l$. Because M_i is a nondeterministic polynomial time machine, l must be polynomial in the size of input, $l = poly(n)$.

For each node $x_i \in \pi$, M_i queries the oracle O and obtains $g(x_i) = y_i$.
Based on these results y_i, M_i takes the decision that π is an accepting path.
In other words, we can view this as if M_i has a function F, which encodes its decision algorithm. The only things M knows after reaching the end of π are: the input he received at the beginning and the oracle's answers $(y_1, y_2, \dots, y_l)$. Therefore, F's output can only depend on these values, so we have that $F(y_1, y_2, \dots, y_l, index, n) = 1$.

Now, what O basically found, is that given the input $(index, n)$ and a series of strings y_i, $i \in \{1, \dots, l\}$, received by M_i in response for the queries for some inputs x_i, $i \in \{1, \dots, l\}$, the path π will be an accepting path and therefore M_i will accept the function on the position $index$.

The adversarial oracle O wants to force M_i to give a wrong answer. All O needs to do is find a *2-to-1* function $\hat{f} : \{0,1\}^n \to \{0,1\}^n$ which will be placed on position $index$ in $\mathcal{F}$ (replace the function g, which was initially set on that position) and which must satisfy the following conditions:

$$\begin{cases} \hat{f}(x_1) = y_1 \\ \hat{f}(x_2) = y_2 \\ \quad \dots \\ \hat{f}(x_l) = y_l \end{cases} \qquad (C_1)$$

In this way, when M_i^O runs the input $(index, n)$ for the function $\hat{f}$, the path π will end in an accepting state because the machine gets exactly the same answers as for the function g which was accepted by π. Thus, M_i^O will decide that $\hat{f}$ is bijective, which is false.

At this point we need to show there exists such a function $\hat{f}$ satisfying (C_1) and we need to properly define it. All the inputs along any path of the computation tree of a nondeterministic machine must be different, thus $(x_1, x_2, \dots, x_l)$ must all be distinct. Because g is bijective this means that $g(x_1)$, $g(x_2), \dots, g(x_l)$ are also distinct.

Therefore, we need to make sure that for the *2-to-1* function $\hat{f}$, there can be no pair (p, q), with $1 \leq p < q \leq l$, such that $\hat{f}(x_p) = \hat{f}(x_q)$.

Hence, we must also set some constraints about the associated *xor mask* string s' of function $\hat{f}$.

We know that $\hat{f}(x_p) = \hat{f}(x_q)$ if and only if $s' = x_p \oplus x_q$.

This means that we must consider all the possible pairs of inputs from path π, (x_p, x_q) and make sure that s' is different from $x_p \oplus x_q$.

The number of all possible distinct strings $c_{p,q} = x_p \oplus x_q$ is at most $\binom{l}{2} = \frac{(l-1)l}{2}$.

M_i is a nondeterministic polynomial-time TM , so every path in the computational tree contains a polynomial in n number of queries, $l \leq p(n)$. Therefore, $\frac{(l-1)l}{2} < 2^n$.

We can choose s_k be any of the 2^n strings in $\{0,1\}^n$ except those $\frac{(l-1)l}{2}$ $c_{p,q}$ strings. (C_2)

Now, since $\hat{f}$ is *2-to-1* with the *xor mask* s' we have $\hat{f}(x')$ = $\hat{f}(x)$ if and only if $x' = x \oplus s'$.

Thus, after we satisfy condition (C_2) we will have all $\hat{f}(x_i)$ distinct from each other, $1 \leq i \leq l$.

When defining a *2-to-1* function $h : \{0,1\}^n \rightarrow \{0,1\}^n$ given a *xor mask* s, it suffices to set the values for half of the inputs ($x_{i_1}, x_{i_2}, \dots, x_{i_{2^{n-1}}}$) such that: $\forall\ 1 \leq a, b \leq 2^{n-1}$ $x_{i_a} \neq x_{i_b} \oplus s$ and $h(x_{i_a}) \neq h(x_{i_b})$. For the other half of the inputs, the values of h are set by the relation:

h($x_{i_j} \oplus s) = h(x_{i_j})\ \forall\ 1 \leq j \leq 2^{n-1}$.

Therefore, in order to define $\hat{f}$, we can set the values $\hat{f}(x_1) = \hat{f}(x_1 \oplus s') = y_1$, …, $f'(x_l) = f'(x_l \oplus s') = y_l$, whereas for the rest of the inputs we can pair them together ($x_{i_j}, x_{i_j} \oplus s'$) and assign to both of them a value from $\{0,1\}^n$ which has not been given to any of the previously assigned inputs.

Consequently, we have constructed a 2-to-1 function $\hat{f}$ satisfying (C_1) and we know that M_i will give a wrong answer for $\hat{f}$ (by accepting it).

We repeat the same procedure for all nondeterministic TM M_i, obtaining that for all of them we can define a *2-to-1* function which the machines will mistakenly identify as *1-to-1*.

OBS **8**. Another aspect we need to take into consideration is that the index specified in the input used to deceive the Turing Machine M_i, should be different from the previously selected indices used to deceive the Turing Machines $M_1, M_2, \dots, M_{i-1}$.

The reason is that if the input to M_i maps to an already set position in $\mathcal{F}$, occupied by some *2-to-1* function f' used to trick the machine M_k, $k < i$, then we will replace f' by a new function f'' such that M_i gives the wrong answer for f''. However, it can be the case that f'' may now be correctly rejected by M_k. Thus, we wouldn't have any more that M_k cannot decide *coSimon*.

In order to avoid this problem, at each step i, we run M_i on the input with index i, and hence we add in $\mathcal{F}$ the resulting *2-to-1* function on position i.

Consequently, $coSimon^O \notin M_i^O$ $\forall i \in \mathbb{N}, M_i \in NP$, or in other words, $\boldsymbol{coSimon^O \notin NP^O}$. □

7.2.2. $NP/poly$-Impossibility

The next step is proving that *coSimon* cannot be solved by a nondeterministic polynomial time algorithm receiving help from a polynomial advice:

MAIN THEOREM 2.3: ***coSimon*** $\boldsymbol{\notin NP/poly^O}$.

Proof. In order to show this, we need to indicate how the adversarial oracle O can deceive any nondeterministic $TM\ M_i$, irrespective of what advice string the machine might receive.

Again, we proceed with a proof by *contradiction*.

Suppose *coSimon* $\in NP/poly^O$.

Then, there exists an Oracle Turing Machine M in $NP/poly$ which returns "yes" when f is bijective and "no" when f is *2-to-1*.

As opposed to the NP case, now M also receives some nonuniform information [18] to help decide if a function is bijective or not. This information is given by an advice function, $h : \mathbb{N} \to \Sigma^*$, such that $|h(n)| \leq p(n)\ \forall n, p$ a polynomial. However, this advice is the same for all inputs of M which have the same length [20, 24].

The input for M is represented by i the index of a function, which is a natural number between 1 and N_n and n the size of the inputs for that function, with n represented in the unary system.
This means that for any function $f_i : \{0,1\}^m \to \{0,1\}^m$ (*1-to-1* or *2-to-1*) we have the same string advice s_{advice} such that $|s_{advice}| \leq p(m + \log(number\ of\ functions\ given\ m))$.
Suppose the number of functions the oracle holds for any length m is $N_m = 2^{q(m)}$, q polynomial. Then, $|s_{advice}| \leq p(m + q(m))$, so $|s_{advice}| = O(p(q(m)))$.

We first present the *intuition of the proof by contradiction*.
Intuitive proof. The number of *2-to-1* and *1-to-1* functions defined on the set $\{0,1\}^n$ is $N_n = 2^{q(n)}$ which is exponential, but we receive the same advice of length polynomial in n, for all these functions. The advice could help the nondeterministic machine by giving for each of these N_n functions some information which can be used in the process of decision.
However, we have the **same** external information of length $p(q(n))$ for any of the $2^{q(n)}$ functions [18]. Because $p(q(n)) \leq 2^{q(n)}\ \forall p, q$ polynomials this would mean that the advice ***cannot even provide one single bit of information*** for each of these functions, necessary to **distinguish** between them.
If instead, the size of the advice would have been equal to N_n, then we would have 1 bit of information assigned for each function. In this case, the advice string could have indicated directly the solution: each bit of advice b_i would be $b_i = 0$ if the function f_i is *2-to-1* and $b_i = 1$ if f_i is bijective and thus the machine could have determined the type of any function.
But in general, if the number of functions is larger than the number of bits of the advice shared between all functions, we cannot obtain any kind of information for each of these functions.

Despite this, the advice string could provide external information [28] for a polynomial number of functions. For example, considering the first $p(q(n))$ functions, the advice could provide one bit of information for each of these functions and therefore, M could decide for them if they are bijective or not. Nonetheless, for the rest of the functions, the NP machine would receive no additional external help to decide whether to accept them or reject.
Therefore, for any length of the input n, we would have $2^{q(n)} - p(q(n))$ functions for which the NP machine M would have no extra information and thus the proof **reduces** to the case of *coSimon* $\notin NP^O$ discussed before. □

We will next present a more ***formal evidence*** for the MAIN THEOREM 2.3.
Formal proof. Suppose there exists a $NP/poly^O\ TM\ M_i$ solving the *coSimon* problem.
The input for M_i is a string Idx encoding the index of the function and the length of the input for that function ($Idx = (index, n)$).

Suppose from now on we will work only with a fixed length n (the results we will obtain can then be generalised for all possible lengths). The maximum index of a function is a number exponential in n, $|index| = q(n)$, q polynomial. Therefore, we obtain that the size of the input for M_i is $O(q(n))$. In this situation, the advice received by M_i for all functions has the property that $|advice| = p(q(n))$, p polynomial. As n is fixed, we can assume that Idx only refers to a function index in $\mathcal{F}$.

To build our proof we will start from some **simplified problems**.

Problem 1. Let's first assume M_i decides the type of the function associated with the input Idx, only based on the advice string and on Idx. In other words, M_i makes no queries to the oracle and the decision procedure for a path can be viewed as: $F(Idx, advice)$.

Solution. We can view the problem in this way: we have no control upon what the advice string might be, but we do control what function to associate for an input Idx such that we deceive M_i.

We build a matrix Mat (Figure 6), where each row represents a possible advice which M_i might receive and each column represents a possible input to M_i. Each cell in Mat, $Mat_{j,k}$ contains a binary value, specifying whether M_i, by receiving the advice adv_j, says that the function on position k is bijective ($Mat_{j,k} = 1$) or *2-to-1* ($Mat_{j,k} = 0$).

Idx / advice	00…0	00…01	…	11…11
00…00	1	0		1
…				
11…11	0	1		0

$2^{p(q(n))}$

$2^{q(n)}$

Figure 6. Example of matrix recording the decisions made by M_i, depending on the advice received

For every input Idx the oracle O chooses a function which **maximizes** the number of wrong answers given by M_i^O.

For the first column, we have $2^{p(q(n))}$ responses, 0 or 1, depending on the advice received by M_i. We count the number of 0("reject") and 1("accept") responses gave by M_i and we obtain u "accept" answers and v "reject" responses, such that $u + v = 2^{p(q(n))}$.

If $u > v$, then for the first position in the set of functions the oracle selects a *2-to-1* function. Otherwise, the oracle selects a *1-to-1* function. In this way, for **more than a half** of the total number of advices, M_i gave the wrong answer, so we can **rule out** these advices. Then, out of the total $2^{p(q(n))}$ advices we now look at only $min(u, v)$ number of advices for the next column, where $min(u, v)$ is obviously less than $\frac{2^{p(q(n))}}{2}$.

And at the second column, we proceed in a similar manner: we look at the responses M_i gave for the remaining advices and consider a function of the other type than the type specified by the majority of answers. This function will be placed on the second position. The number of advices left is now less than $\frac{2^{p(q(n))}}{2^2}$. In this manner, for every column we **reduce** with **more than a half** the number of advices which could be correct.

Therefore, at the $p(q(n)) + 1$ column of Mat we will definitely be left with no correct advice and we conclude that there is no polynomial advice which could help M_i solve this problem. □

Problem 2. Now, we move to a second problem, specifically, showing that $\boldsymbol{coSimon \notin P/poly}$.

Solution. We use a similar demonstration based on the construction of a matrix recording all the possibilities for the behaviour of a $P/poly\ TM\ M_i$ which would be able to solve *coSimon*. For this case when M_i is a deterministic polynomial time machine, we know that it can only query the oracle O for a polynomial number of times [17, 20].

Suppose M_i queries O at most k times (k polynomial in n), with all the inputs queried distinct from one another. These queries can be any strings of length n, and we name them : $x_1, x_2, \dots, x_k$.

We build a new matrix $\boldsymbol{Mat}$' (Figure 7). The rows of Mat' represent the possible advices which M_i could have received and the possible k inputs queried by M_i. The columns represent the function indices and the answers given by O in response to the k queries. The values in each cell of $Mat'_{j,index}$ specify whether M_i receiving a particular advice and a particular set of responses from the oracle, accepts or not the function positioned at $index$.

If M_i would have not received any external advice, the oracle O would know its algorithm and so would know the queries the machine is going to make [22]. However, an advice can change the behaviour of M_i and determine a new sequence of queries requested by M_i.

Thus, for every possible external information, we have to consider a **new behaviour** of M_i, each of them can be seen as a **new different deterministic polynomial** time TM [21]. Since each such deterministic polynomial-time TM is identified by a sequence of at most k queries, we only need to take into account **every possible array of $\boldsymbol{k}$ different queries**.

Hence, each advice adv_i corresponds to a sequence $(x_{i,1}, x_{i,2}, \dots, x_{i,k})$ (which we call ***k-sequence)***, $x_{i,j} \in \{0,1\}^n\ \forall j \in \{1,\dots,k\}$, $\forall i \in \{1,\dots,2^{p(q(n))}\}$. We name ***k-answer,*** the array of the k answers given by the oracle in response to a k-sequence. Our goal is that, for every function index, to eliminate a fraction of these possible k-*sequences* such that, for the last index we will be left with no possible k-*sequence*. Then, we conclude that no advice can help a P machine solve the *coSimon* problem.

Index function query answers / advice/ k-queries		Index function = 1			…	Index function = $2^{q(n)}$		
		$(x_{1,1}, \dots, x_{1,k})$	…	$(x_{R,1}, \dots, x_{R,k})$	…	$(x_{1,1}, \dots, x_{1,k})$	…	$(x_{R,1}, \dots, x_{R,k})$
adv_1	$(x_{1,1}, \dots, x_{1,k})$	1	…	0	…	1	…	0
adv_2	$(x_{2,1}, \dots, x_{2,k})$	0	…	0	…	-	…	1
…			…				…	
$adv_{2^{p(q(}}$	$(x_{2^{p(q(n))},1}, \dots, x_{2^{p(q(n))},k})$	-	…	1	…	1	…	0

Figure 7. Example of matrix for the P/poly case

Every response to a query is a number between 0 and $2^n - 1$. Thus, the number of possible k-answers (containing k distinct answers) given in response to a k-sequence asked by M_i is equal to the number of arrangements of 2^n answers taken k at a time, $P(2^n, k)$. However, the function may be *2-to-1,* so a k-answer must not always contain k distinct answers (we need to also allow for some pairs from the k values to be equal). We could consider any possible *xor mask* for f and determine for which inputs (x_i, x_j) we can have $f(x_i) = f(x_j)$. One solution might be listing all possible 2^n values for the elements of a k-answer and then we invalidate the entries in the matrix which cannot be a subset of the set of values of any bijective or 2-to-1 function. For example, we could insert in those invalid corresponding cells the symbol "- ").

Therefore, the number of columns of Mat' is V, where $2^{q(n)} \cdot P(2^n, k) < V < 2^{q(n)} \cdot 2^{nk}$.

Unlike the previous problem, for each function index we now have a *block* of columns (instead of just one column) and every column represents a possible k-answer.

Consequently, for function $index = i$ we initially set a *1-to-1* function f. We then look at the results provided by M_i (accept/reject) given each of the advices available at this stage. We count the total number of accepts and rejects. If the number of rejects is greater than the number of accepts, then it means that more than a half of the advices are *wrong* and we can eliminate them and move to the next step (function index $i + 1$).

Otherwise, we can proceed in the following way. We select half of the inputs, for instance the first 2^{n-1} inputs, $S = \{0, 1, \dots, 2^{n-1} - 1\}$ and construct a *2-to-1* function g, such that $g(0) = f(0)$, $g(1) = f(1), \dots, g(2^{n-1} - 1) = f(2^{n-1} - 1)$.

We can build such a function g, by setting its associated *xor mask* s_g such that $s_g \neq u \oplus v$ $\forall u, v \in \{0, \dots, 2^{n-1} - 1\}$, $u \neq v$. Such a string s_g exists, because we notice that all the values of $u \oplus v$ are less than $2^{n-1} - 1$, so, for example, we can set $s_g = 2^{n-1}$.

We now consider the *majority* of advices adv_j, the ones for which the resulting deterministic polynomial time TMs are saying that the function with the $index = i$ is bijective. Each of these TMs corresponds to a k-sequence and suppose the number of these machines is L (we know that $L > \frac{1}{2} \cdot number\ of\ possible\ advices$). Now, in the worst case, each advice adv_j corresponds to a different k-sequence, meaning that $L = \binom{2^n}{k}$. This situation is possible because we *cannot control* the exact size of the advice, we just know that the external advice must be polynomial in the size of input. Thus, in the worst case, we can have $|advice| = \log \binom{2^n}{k}$, which is a polynomial function in n.

By selecting only the deterministic TMs whose queries consist of inputs from the set S, we have that all these TMs will accept function g, as they get the same query responses as for the f function. But, since g is *2-to-1,* they are all wrong and therefore, we can eliminate the lines of the matrix corresponding to these advices. The number of k-sequences consisting of inputs only from the set S is $\binom{2^{n-1}}{k}$, so at each step we can eliminate at least $\binom{2^{n-1}}{k}$ lines of the matrix Mat'.

We can further improve the number of eliminations by making the next **essential observation**.

OBS **8**. We can additionally eliminate k-sequences which also contain inputs x' from the other half of inputs ($\{0, 1\}^n - S$) as long as $x' \neq x \oplus s_g$ $\forall x \in S$. This is because we are assuming that each of the remaining TMs can solve the *coSimon* problem, therefore, they always accept correctly a bijective function. In other words, if the oracle returns any k values which could be the values of any bijective function (not only from f), then these TMs will accept.

Thus, the only condition the oracle must satisfy in order to deceive a deterministic TM, is that for **k distinct inputs to return k distinct values**. For this reason, we now take into account any k-sequence for which the fixed *2-to-1* function g returns distinct values.

Therefore, the number of k-sequences we can select is: $E = \binom{2^{n-1}}{0} \cdot \binom{2^{n-1}}{k} + \binom{2^{n-1}}{1} \cdot \binom{2^{n-1}-1}{k-1} + \dots + \binom{2^{n-1}}{k} \cdot \binom{2^{n-1}-k}{0} = \sum_{i=0}^{k} \binom{2^{n-1}}{i} \cdot \binom{2^{n-1}-i}{k-i}$

We further obtain that $E = \sum_{i=0}^{k} \frac{(2^{n-1})!}{i! \cdot (2^{n-1}-i)!} \cdot \frac{(2^{n-1}-i)!}{(k-i)! \cdot (2^{n-1}-k)!} = \frac{(2^{n-1})!}{(2^{n-1}-k)!} \cdot \sum_{i=0}^{k} \frac{1}{i! \cdot (k-i)!} = \frac{(2^{n-1})!}{(2^{n-1}-k)!} \cdot \frac{1}{k!} \cdot \sum_{i=0}^{k} \frac{k!}{i! \cdot (k-i)!} = \frac{(2^{n-1})!}{(2^{n-1}-k)!} \cdot \frac{1}{k!} \cdot \sum_{i=0}^{k} \binom{k}{i} = \frac{(2^{n-1})!}{(2^{n-1}-k)!} \cdot \frac{1}{k!} \cdot 2^k = \binom{2^{n-1}}{k} \cdot 2^k$.

For all these E queries the resulting TMs will give the wrong answer (saying that g is bijective), so we can eliminate E lines of the matrix Mat' in one step.
We can see that we are now left with queries containing inputs both from S and from $\{0,1\}^n - S$, such that we have at least one pair (x, x'), where $x \in S, x' \in \{0,1\}^n - S$ and $x' = x \oplus s$.

Then, we can repeat the whole procedure for all $2^{q(n)}$ function indices. We obtain that we are able to eliminate in the worst case at least $2^{q(n)} \cdot \binom{2^{n-1}}{k} \cdot 2^k$ advices. As $2^{q(n)} \cdot \binom{2^{n-1}}{k} \cdot 2^k > \binom{2^n}{k}$, this means that we have **discarded all lines of Mat'**, so no possible polynomial advice could help any deterministic machine solve *coSimon*. Therefore, ***coSimon* $\notin$ *P/poly*.** □

Using the proof techniques used in ***Problem 1*** and ***Problem 2***, we can finally proceed to demonstrate MAIN THEOREM 2.3.

Suppose $\exists M_i \in NP/poly$ solving *coSimon*.

Once again, we need to take into consideration every possible advice received by M_i. As in the previous problems, the number of advices is $2^{p(q(n))}$, where p and q are 2 polynomials.

For each of these possible advice strings which M_i might receive, we obtain a **new nondeterministic polynomial-time machine $M_{i,j}$** [17, 20] and we must demonstrate that none of these resulting machines is able to solve our *coSimon*.

Following the same strategy, we build a new matrix $\boldsymbol{Mat''}$ (Figure 8), where each line corresponds to a possible advice (the resulting NP machine) and for each possible function index we have a block of columns corresponding to possible k-answers returned by the oracle O.

Index function / query answers — advice/resulted NP machine		Index function = 1			…	Index function = $2^{q(n)}$		
		$(x_{1,1}, \dots, x_{1,M})$	…	$(x_{R,1}, \dots, x_{R,M})$	…	$(x_{1,1}, \dots, x_{1,M})$	…	$(x_{R,1}, \dots, x_{R,M})$
adv_1	$M_{i,1}$	1	…	0	…	1	…	0
adv_2	$M_{i,2}$	0	…	0	…	0	…	1
…			…		…		…	
$adv_{2^{p(q(n))}}$	$M_{i,2^{p(q(n))}}$	1	…	1	…	1	…	0

Figure 8. Example of matrix for the P/poly case

For the first step (function $index = 1$), we choose a bijective function $f : \{0,1\}^n \rightarrow \{0,1\}^n$.

In this way, we fixed the answers the oracle will return to each $M_{i,j}$. This results in a single column for $index = 1$, no matter what the computational trees of each $M_{i,j}$ would look like. Each of the values in this column will be a 0 or 1 specifying if $M_{i,j}$ accepts or not this f. We then iterate through all possible advices and look at the answers gave by each machine $M_{i,j}$.

We can count the number of accepts and rejects. If the number of rejects is larger, this means that the more than a half of the TMs are wrong and we can eliminate this majority for the following step.

Now, consider the case when the number of "yes" answers is in majority. Suppose the number of these machines is $L, L > \frac{2^{p(q(n))}}{2}$. Then, we want to show that we can eliminate at each stage a fraction of these L lines, such that at the final step $index = 2^{q(n)}$, we are left with no possible nondeterministic machines capable of solving our problem. If we can eliminate at each step a constant fraction of lines c, then at step i we would have $2^{p(q(n))} \cdot (1-c)^i$ remaining lines and until we reach the last index function we would eliminate all advices.

From the selected *1-to-1* function f we want to build a *2-to-1* function $\hat{f}$ in the following manner:

We choose the first 2^{n-1} inputs $S = \{0, 1, \ldots, 2^{n-1}\}$ and construct a *2-to-1* function $\hat{f}$ with the *xor mask* s , such that $\hat{f}\,(0) = f(0)$, $\hat{f}\,(1) = f(1)$, ..., $\hat{f}\,(2^{n-1}) = f(2^{n-1})$ and $s \neq u \oplus v$ $\forall u, v \in \{0, \ldots, 2^{n-1} - 1\}$, $u \neq v$ (for instance s = 2^{n-1}).

Each of the L machines is incorrectly accepting f. Because they are nondeterministic polynomial-time machines, this fact is equivalent to saying that: for any of these TMs, there exists a path containing a polynomial number of queries, such that the path ends in an accepting state [20, 33].

Therefore, for each advice adv_j we can associate an accepting path π_j of the machine $M_{i,j}$. We suppose each of these paths contains a sequence of at most k queries to the oracle, where k is polynomial in n. Hence, if we assume that for a given advice, the correspondent machine $M_{i,j}$ always accepts correctly a bijective function, then by giving the answers to the k-sequence as coming from any bijective function, $M_{i,j}$ must accept the given function.

We pick an accepting path for each of these nondeterministic machines. Then, we can deceive a path π_j by returning in response to the queries along π_j, a k-answer consisting of distinct values coming from a *2-to-1* function. We can see each of these **accepting paths $\boldsymbol{\pi_j}$ as a deterministic polynomial time machine**. Then, we deduce that we only need to consider all possible k-sequences and eliminate them until the last step (function $index = 2^{q(n)}$). Consequently, the **proof reduces to** the ***Problem 2*** of showing that no $P/poly$ machine can solve the *coSimon* problem, so we conclude that:

$\boldsymbol{coSimon \notin NP/poly}$ □

In this section we have described the problem *coSimon* solvable by a BQP machine and we presented both an *intuitive* and a *formal* proof that *coSimon* $\notin NP/poly^O$. Using the fact that *coSimon* $\in BQP^O$ and that any problem in BQP^O can be reduced in polynomial time to a problem G from $BQP - Hard^O$ [17], we can derive that no $BQP - Hard^O$ problem can be solved by a machine from the class $NP/poly^O$.

In the next part, we will present and analyse another problem named *Forrelation* [25], which suggests the gap between the complexity classes BQP and $NP/poly$.

8. Forrelation Problem

Another interesting problem we study in order to indicate the separation between the BQP and the $NP/poly$ classes is called the *Forrelation* Problem [25]. One reason why we delve into this problem is that it proves the *highest gap* known so far between classical and quantum complexity.

DEFINITION 9: The *Forrelation* problem determines whether one Boolean function $f : \{0,1\}^n \rightarrow \{-1,1\}$ is *highly correlated* with the Fourier transform $\tilde{g}$ of a second Boolean function $g : \{0,1\}^n \rightarrow \{-1,1\}$, given the promise that f and $\tilde{g}$ are either *highly correlated* or *strongly uncorrelated*.

Specifically, the level of correlation between f and $\tilde{g}$ is measured by the quantity:

$\boldsymbol{\Theta_{f,g}} = \frac{1}{2^{\frac{3n}{2}}} \cdot \sum_{x,y \in \{0,1\}^n} (-1)^{x \cdot y} f(x) g(y)$. The *Forrelation* problem tells us that either $|\boldsymbol{\Theta_{f,g}}| < \frac{\mathbf{1}}{\mathbf{100}}$ - corresponding to a high correlation or $\boldsymbol{\Theta_{f,g}} \geq \frac{\mathbf{3}}{\mathbf{5}}$ – corresponding to a large uncorrelation.

As in the case of *Simon's* problem, we do not have direct access to the table values of the 2 functions, but we have an oracle O which given an input can return the values of the 2 functions in that point.

While classically we would need at least $\frac{2^{n-1}}{n}$ queries to solve the problem [25, 30], we will demonstrate that in the quantum paradigm *Forrelation* can be solved using only *1 query* to the oracle.

8.1. Quantum Algorithm

THEOREM 3.1. $Forrelation\ Problem \in BQP^O$

Proof. The *Forrelation* problem can be solved using the quantum circuit in Figure 9:

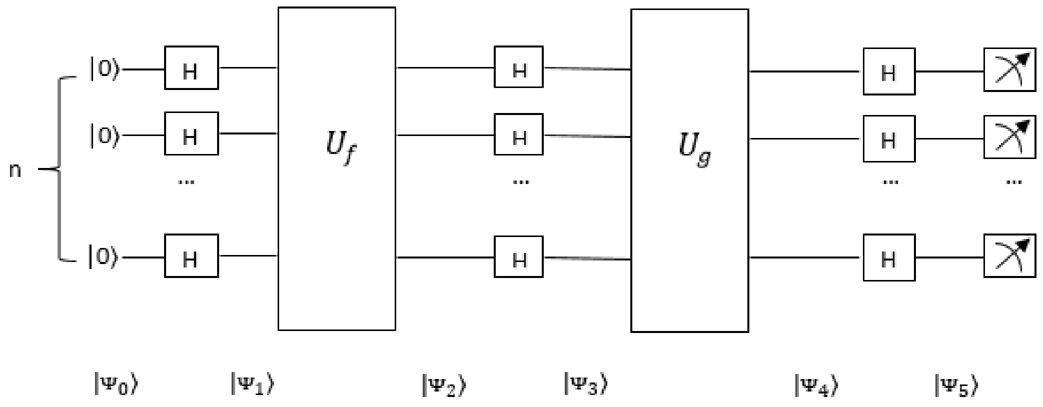


Figure 9. Circuit for solving the Forrelation problem

The initial state is $|\Psi_0\rangle = |0\rangle^{\otimes \mathrm{n}}$. After we apply n Hadamard gates we reach the state [1]:

$|\Psi_1\rangle = \frac{1}{(\sqrt{2})^n} \cdot \ (|0\rangle + |1\rangle)^n = \frac{1}{(\sqrt{2})^n} \cdot \sum_{x \in \{0,1\}^n} |x\rangle.$

Next, we execute the unitary for the f function [29], which has the following effect:

$$|x\rangle \xrightarrow{U_f} f(x) \cdot |x\rangle$$

Therefore, we obtain $|\Psi_2\rangle = \frac{1}{(\sqrt{2})^n} \cdot \sum_{x \in \{0,1\}^n} f(x) \cdot |x\rangle$

The second round of Hadamards, will transform this state to:

$$|\Psi_3\rangle = \frac{1}{(\sqrt{2})^n} \cdot \frac{1}{(\sqrt{2})^n} \cdot \sum_{x \in \{0,1\}^n} \left(\sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} \cdot f(x) \cdot |y\rangle \right)$$

Following this, we apply the second unitary for the g function, U_g, obtaining the state $|\Psi_4\rangle$:

$$|\Psi_4\rangle = \frac{1}{2^n} \cdot \sum_{x \in \{0,1\}^n} \left(\sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} \cdot f(x) \cdot f(y) \cdot |y\rangle \right)$$

Finally, the last set of Hadamard gates will lead us to the state $|\Psi_5\rangle$:

$$|\Psi_5\rangle = \frac{1}{2^n} \cdot \frac{1}{(\sqrt{2})^n} \cdot \sum_{x \in \{0,1\}^n} \left(\sum_{y \in \{0,1\}^n} \sum_{z \in \{0,1\}^n} (-1)^{y \cdot z} \cdot (-1)^{x \cdot y} \cdot f(x) \cdot g(y) \cdot |z\rangle \right)$$

Now, we measure the probability of getting $|0\rangle^{\otimes n}$ [12].

Using the projector $P_0 = |0\rangle^{\otimes n} \langle 0|^{\otimes n}$, we obtain the probability equals to $P(0) = \langle \Psi_5 | {P_0}^{+} P_0 || \Psi_5 \rangle$

$$P(0) = \left[\frac{1}{2^{\frac{3n}{2}}} \cdot \sum_{x,y \in \{0,1\}^n} (-1)^{x \cdot y} f(x) g(y) \right] \cdot \left[\frac{1}{2^{\frac{3n}{2}}} \cdot \sum_{x,y \in \{0,1\}^n} (-1)^{x \cdot y} f(x) g(y) \right].$$

The amplitude associated with the state $|0\rangle^{\otimes n}$ is $A = \sqrt{P(0)} = \frac{1}{2^{\frac{3n}{2}}} \cdot \sum_{x,y \in \{0,1\}^n} (-1)^{x \cdot y} f(x) g(y)$, which is exactly the quantity $\Theta_{f,g}$ we needed.

Moreover, we observe that we obtained $\Theta_{f,g}$ with a single query to the oracle for f and a single query to the oracle for g. Consequently, we have described a BQP algorithm solving the *Forrelation* problem.

□

8.2. k-Forrelation

We create a generalization of this problem, called k-*Forrelation* [25] which involves k Boolean functions. We choose to discuss about this problem because it has the important property of being a $BQP - complete$ problem [17, 19].

DEFINITION 10. In the k-*Forrelation* we have the functions $f_1, f_2, \dots, f_k$ with $f_i: \{0,1\}^n \to \{-1, 1\}$, $\forall i \in \{1, \dots, k\}$. We compute the value of:

$$\Theta_{f_1, f_2, \dots, f_k} = \frac{1}{2^{\frac{(k+1)n}{2}}} \cdot \sum_{x_1, x_2, \dots, x_k \in \{0,1\}^n} (-1)^{x_1 \cdot x_2} \cdot (-1)^{x_2 \cdot x_3} \cdot \dots \cdot (-1)^{x_{k-1} \cdot x_k} \cdot f_1(x_1) \cdot \dots \cdot f_k(x_k)$$

and need to decide whether $\boldsymbol{\Theta_{f_1, f_2, \dots, f_k} \geq \frac{3}{5}}$ or $\boldsymbol{|\Theta_{f_1, f_2, \dots, f_k}| \leq \frac{1}{100}}$.

To prove that k-*Forrelation* is a $BQP - complete$ problem, we need to show that it is both a BQP and a $BQP - hard$ problem [20].

THEOREM 3.1.1: k-*Forrelation* ∊ BQP^O

Proof. We can follow a generalisation of the proof for THEOREM 3.1 (Fig. 10).

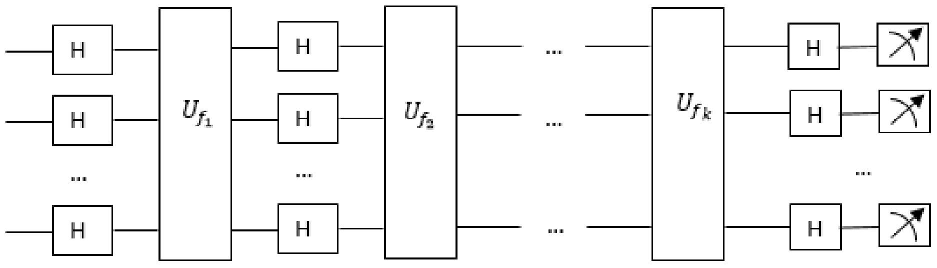


Figure 10. C_{Forr} Circuit for k-Forrelation

As in the case of the *Forrelation* problem, the value of quantity we are interested in, $\Theta_{f_1,f_2,\dots,f_k}$, is represented by the amplitude corresponding to the state $|0\rangle^{\otimes n}$ from the final superposition quantum state resulted from the above C_{Forr} circuit [25].

We can see that we only need to make one query to the oracles for each of the k functions, so the k-*Forrelation* problem can be solved using k quantum queries. □

THEOREM 3.1.2: k-*Forrelation* ∊ $BQP - Hard^O$.

Proof. In order to show that it is a $BQP - hard$ problem, we use a **polynomial-time reduction** [32] from a problem known to be in $BQP - complete$: the *QSIM* problem [30]. In *QSIM* we receive as input a quantum circuit Q containing only *Hadamard* and $CSIGN$ quantum gates. The problem asks us to decide whether the probability of Q ending in the state $|0\rangle^{\otimes n}$ resides in one of 2 possible intervals. To be more specific, given that for amplitude corresponding to this probability $S_Q = \langle 0|^{\otimes n} Q |0\rangle^{\otimes n}$ we have that either $|S_Q| \le \frac{1}{100}$ or $S_Q \ge \frac{3}{5}$, the *QSIM* problem requires us to identify which of the 2 cases holds for the quantum circuit Q.

We notice that both S_Q in the *QSIM* problem and $\Theta_{f_1,f_2,\dots,f_k}$ in the k-*Forrelation* are promised to be in the exact same intervals [25, 30]. Therefore, a reduction from *QSIM* to k-*Forrelation* entails showing that for any circuit Q consisting of *Hadamard* or $CCSIGN$ gates we are able to find k Boolean functions $f_1, f_2, \dots, f_k$ such that $S_Q = \Theta_{f_1,f_2,\dots,f_k}$.

As seen in Figure 10, $\Theta_{f_1,f_2,\dots,f_k}$ is obtained from a circuit (C_{Forr}) which contains k blocks of gates, where each block has a *Hadamard* gate applied on every qubit, followed by a unitary operator for a function.

Our reduction implies that this circuit C_{Forr} must match any possible circuit consisting of *Hadamard* and $CCSIGN$ gates only. C_{Forr} already contains *Hadamard* gates, thus, we must use the function operators U_{f_i} to simulate $CCSIGN$ gates [1].

We remind that the $CCSIGN$ unitary takes as input 3 qubits $|x\rangle|y\rangle|z\rangle$ and returns $(-1)^{x \cdot y \cdot z}|x\rangle|y\rangle|z\rangle$.

Suppose Q receives an n qubits input and contains a $CCSIGN$ gate applied on the qubits x_i, x_j and x_k. Then, we can simulate Q by selecting one of our k functions to be $f_l(x_1, x_2, .., x_n) = (-1)^{x_i \cdot x_j \cdot x_k}$.

Therefore, f_l affects only the 3 qubits and leaves the rest of them unchanged.

However, as seen in C_{Forr}, any unitary U_{f_i} is also surrounded by n *Hadamards* to the left and n to the right. If we just want to apply a $CCSIGN$ on 3 qubits and have no *Hadamards* applied before or after, then we can set the functions f_{l-1} and f_{l+1} to be constant: f_{l-1} = $f_{l+1} = 1$. If $f_{l-1} = 1$, then for f_{l-1} we have: a block containing n *Hadamards* followed by the unitary for f_{l-1}, which is now the identity matrix, followed by another n *Hadamards*. Thus, the structure for f_{l-1} reduces to a pair of *Hadamards* applied to each qubit. But since we know that $H^2 = I$ [29], all these *Hadamards* cancel each other. The same process takes place for f_{l+1}, so we are only left with the $CCSIGN$ gate obtained by f_l.

So far we treated the cases when we apply *Hadamards* on all n qubits or we cancel all *Hadamards*. Now, we must find a solution when in Q there are H gates applied only to a subset of the n qubits.

Suppose we want to apply a *Hadamard* to only 2 qubits x_i, x_j, without affecting the rest of the qubits.

In this case we add 3 functions: f_l, f_{l+1}, f_{l+2}, such that:

$f_l(x_1, x_2, \dots, x_n)$ = $f_{l+1}(x_1, x_2, \dots, x_n)$ = f_{l+2} $(x_1, x_2, \dots, x_n)$ = $(-1)^{x_i \cdot x_j}$

Then, this would result in the following fragment of the C_{Forr} circuit [25] (Figure 11).

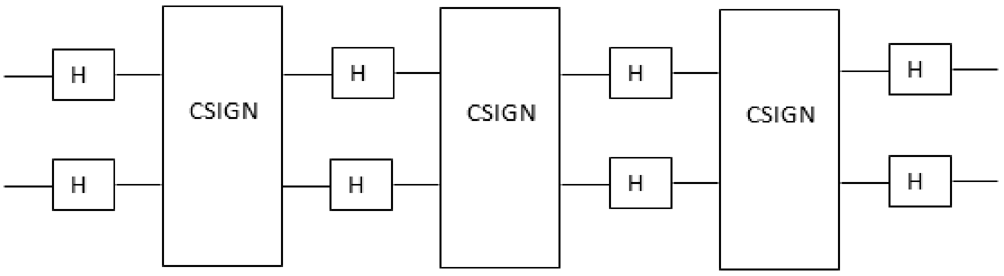


Figure 11. Fragment of Forrelation circuit to simulate Hadamard on 2 qubits

The result of this circuit is the application on the qubits x_i, x_j of the operator:

$U = H_2 \cdot CSIGN \cdot H_2 \cdot CSIGN \cdot H_2 \cdot CSIGN \cdot H_2$

We use the relation that $(H_2 \cdot CSIGN)^3 = SWAP$ [3, 12]. Therefore, we have $U = SWAP \cdot H_2$.

On the other hand, for the other $n - 2$ qubits (different from x_i, x_j), the 3 functions f_l, f_{l+1}, f_{l+2} do not alter them, so we only apply the *Hadamards* on those $n - 2$ qubits, resulting the following operator acting on them:

$U' = H \cdot I \cdot H \cdot I \cdot H \cdot I \cdot H = H^4 = I$,

which means that the other $n - 2$ qubits remain unchanged as we wanted.

However, we saw that the operator U acting on x_i, x_j has besides the *Hadamard,* an additional operator, $SWAP$. But, since we are aware of this fact, all we need to do is to swap the 2 qubits x_i, x_j every time we want to apply a *Hadamard* on them.

Now, we can generalize our result for the application of *Hadamard* gates on a subset of qubits, of size larger than 2. Consider we want H gates acting on m qubits. Then, if m is even we can group these qubits into $\frac{m}{2}$ pairs and use for each pair the procedure presented above (Figure 11) [25]. If instead, m is odd, then we can add an extra qubit in the circuit, so that we can group again all qubits into pairs and repeat the same algorithm.

The last remaining problem occurs in the case where in Q we have 2 consecutive $CCSIGN$ gates applied on the same qubits (call them x, y, z). In this situation, we would need 2 functions equal to $(-1)^{x \cdot y \cdot z}$. To get rid of the *Hadamards* in between the $CCSIGN$ operators, we insert another function f', such that $f_j = (-1)^{x \cdot y \cdot z}$, $f_{j+1} = f'$, $f_{j+2} = (-1)^{x \cdot y \cdot z}$ and $f' = 1$. Consequently, the *Hadamards* between f_j and f_{j+1} will be cancelled by the *Hadamards* between f_{j+1} and f_{j+2} and we will remain with 2 consecutive $CCSIGN$ gates [1, 29].

All in all, we simulated each $CCSIGN$ gate of the circuit Q using 1 function in the *Forrelation* problem and each H gate using 3 functions.

Therefore, starting from a quantum circuit Q containing M gates: *Hadamards* or $CCSIGN$, we are able to find k functions, $f_1, f_2, \dots, f_k$ such that the *Forrelation* circuit simulates the circuit of Q ($S_Q = \Theta_{f_1, f_2, \dots, f_k}$), where $k = O(M)$). Moreover, we obtained that any of the k functions is of the form $f_i : \{0,1\}^{n+1} \to \{-1, 1\}$ $\quad f_i(x_1, x_2, \dots, x_{n+1}) = (-1)^{x_a \cdot x_b}$ or $f_i(x_1, x_2, \dots, x_{n+1}) = (-1)^{x_u \cdot x_v \cdot x_w}$ (the input has length $n + 1$, because we might need to add an extra bit to solve the *Hadamard* issue described above).

Then, we have found a reduction in polynomial time from a $BQP - complete$ problem (*QSIM*) to *k-Forrelation*, when $k = O(n)$. Additionally, we have given a BQP algorithm to this problem, thus, we have shown **that *k-Forrelation* is in $BQP - complete$**. $\square$

At this point, we could not find a way to relate the *k-Forrelation* to the class $NP/poly$. Instead, we are now going to link this $BQP - complete$ problem to a similar oracle problem [25, 30] which is easier to work on.

8.3. Real Forrelation

DEFINITION 11. In the *Real Forrelation* problem [25] we are given 2 functions $f, g : \{0,1\}^n \to \mathbb{R}$ and we can obtain values of these functions in different points through an oracle O. Given the fact that there are possible 2 scenarios involving the 2 input functions, we are asked to determine in which of the 2 cases we are. The 2 scenarios are the next ones:

i. For all $x, y \in \{0,1\}^n$, $f(x)$ and $g(y)$ are both drawn from normal distributions with mean 0 and variance 1
ii. For all $x \in \{0,1\}^n$, $f(x)$ are drawn from the normal distribution, but function g depends on the values of f: $g(x) = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} f(y) \cdot (-1)^{x \cdot y}$

The second case can be rewritten as: $g = H \cdot f$, where we considered f and g as 2 vectors containing the values of the functions and H is the *Hadamard* matrix [4].

What we want to show is the following essential property [25]:

THEOREM 3.2.1. ***Real Forrelation* is at most as difficult to solve as the *Forrelation* problem**. Then, by showing that we can obtain a negative answer for *Real Forrelation* being solved by a TM in $NP/poly$, this would imply the same negative result for *Forrelation*.

Proof. Showing that *Forrelation* is at least as difficult as *Real Forrelation* can be achieved by finding a **polynomial reduction** from *Real Forrelation* to *Forrelation* (*Real Forrelation*$\leq_{poly}$*Forrelation*) [20].

Given an input for the *Real Forrelation* problem, consisting of 2 real functions, we need to create an instance of the *Forrelation* problem.

In other words, from $f, g : \{0,1\}^n \to \mathbb{R}$ we need to create 2 Boolean functions $F, G : \{0,1\}^n \to \{-1,1\}$ such that if f, g are *uncorrelated* then we have $|\Theta_{F,G}| \leq \frac{1}{100}$ and if $g = Hf$ we have $\Theta_{F,G} \geq \frac{3}{5}$.

We build F and G in the following way:

$F(x) = sign(f(x))\ \forall x \in \{0,1\}^n$

$G(x) = sign(g(x))\ \forall x \in \{0,1\}^n$

Now, given that $g = Hf$, we need to show that $|\Theta_{F,G}| \leq \frac{1}{100}$

$\Theta_{F,G} = \frac{1}{2^{\frac{3n}{2}}} \cdot \sum_{x,y \in \{0,1\}^n} (-1)^{x \cdot y} F(x)G(y)$

$g(x) = (Hf)_x = (-1)^{x \cdot 0} \cdot f(0) + (-1)^{x \cdot 1} \cdot f(1) + \ldots + (-1)^{x \cdot (2^n - 1)} \cdot f(2^n - 1) = \sum_{i=0}^{2^n-1} (-1)^{x \cdot i} \cdot f(i)$

We compute the ***estimated value*** [34] of each term of $\Theta_{F,G}$:

$T_{x,y} = E[(-1)^{x \cdot y} \cdot F(x) \cdot G(y)] = E[(-1)^{x \cdot y} \cdot sign(f(x)) \cdot sign(\sum_{i=0}^{2^n-1} (-1)^{y \cdot i} \cdot f(i))]$

We can rewrite $T_{x,y}$ as $T_{x,y} = E\left[(-1)^{x \cdot y} \cdot sign(f(x)) \cdot sign(C + (-1)^{y \cdot x} \cdot f(x))\right]$, where

$C = \sum_{\substack{i=0 \\ i \neq x}}^{2^n-1} (-1)^{y \cdot i} \cdot f(i)$.

Because we know that all values of f are independently drawn from $\mathcal{N}(0,1)$, we have that $f(x)$ and C are also independent [33, 34] (as C involves every value of f except the one on input x) and hence:

$E\left[(-1)^{x \cdot y} \cdot sign(f(x)) \cdot sign(C)\right] = 0$

Now, when we add $(-1)^{y \cdot x} \cdot f(x)$ to C there are 2 possibilities:

a) $sign(C + (-1)^{y \cdot x} \cdot f(x)) = sign(C)$, so $T_{x,y} = E\left[(-1)^{x \cdot y} \cdot sign(f(x)) \cdot sign(C)\right] = 0$

b) $sign(C + (-1)^{y \cdot x} \cdot f(x)) = sign((-1)^{y \cdot x} \cdot f(x))$ and $sign((-1)^{y \cdot x} \cdot f(x)) \neq sign(C)$

Therefore, we obtain that $T_{x,y} = 2 \cdot P\left(sign(C + (-1)^{y \cdot x} \cdot f(x)) \neq sign(C)\right)$

$sign(C + (-1)^{y \cdot x} \cdot f(x)) \neq sign(C)$ can only happen when:

$sign(C) \neq sign((-1)^{y \cdot x} \cdot f(x))\ and\ |C| < |(-1)^{y \cdot x} \cdot f(x)|$.

Consider $C(t)$ the continuous random variable of C [34]. Then, we have that:

$T_{x,y} = 2 \cdot \int_{-\infty}^{+\infty} P(|t| < |(-1)^{y \cdot x} \cdot f(x)|) \cdot C(t)dt = 4 \cdot \int_{0}^{+\infty} P(t < (-1)^{y \cdot x} \cdot f(x)) \cdot C(t)dt$

Since $C = \sum_{\substack{i=0 \\ i \neq x}}^{2^n-1} (-1)^{y \cdot i} \cdot f(i)$ and each $f(i)$ represents a Gaussian $\mathcal{N}(0,1)$, we obtain that C is also a Gaussian with mean 0 and variance $2^n - 1$, namely that $C(t) = \frac{1}{\sqrt{2\pi(2^n - 1)}} \cdot e^{-\frac{x^2}{2(2^n - 1)}}$.

Moreover, due to the fact that the mean of f is 0 and that $(-1)^{y \cdot x}$ can only take values -1 or 1 it suffices to consider the case $(-1)^{y \cdot x} = 1$.

Hence, $T_{x,y} = 4 \cdot \int_0^{+\infty} P\big(t < f(x)\big) \cdot \frac{1}{\sqrt{2\pi(2^n - 1)}} \cdot e^{-\frac{t^2}{2(2^n-1)}}\, dt$

Using the fact that the function $e^{-\frac{t^2}{2(2^n-1)}}$ is *strictly decreasing* on the interval $(0, \infty)$, then $e^{-\frac{t^2}{2(2^n-1)}} < 1$ and we get:

$$T_{x,y} \le \frac{4}{\sqrt{2\pi(2^n - 1)}} \cdot \int_0^{+\infty} P\big(t < f(x)\big)\, dt$$

But, $\int_0^{+\infty} P\big(t < f(x)\big)\, dt = E(f(x))$

As any value $f(x)$ is an independent $\mathcal{N}(0,1)$ its probability density function [33] is equal to $\frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$

Thus, $E(f(x)) = \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{t^2}{2}} \cdot t\, dt = \frac{1}{\sqrt{2\pi}} \cdot \left(-e^{-\frac{t^2}{2}}\right) \Big|_0^{\infty} = \frac{1}{\sqrt{2\pi}} \cdot (1 - e^{-\infty}) = \frac{1}{\sqrt{2\pi}}$

Therefore, $T_{x,y} \le \frac{4}{\sqrt{2\pi(2^n - 1)}} \cdot \frac{1}{\sqrt{2\pi}} = \frac{2}{\pi\sqrt{2^n - 1}} \le \frac{2}{\pi\sqrt{2^n}} + O\left(\frac{1}{2^{\frac{3n}{2}}}\right) \qquad \forall x, y \in \{0,1\}^n$

On the other hand, for any constant $D > 0$ we have that:

$$T_{x,y} = \frac{4}{\sqrt{2\pi(2^n - 1)}} \cdot \Big[\int_0^D P\big(t < f(x)\big) \cdot e^{-\frac{t^2}{2(2^n-1)}}\, dt + \underbrace{\int_D^{+\infty} P\big(t < f(x)\big) \cdot e^{-\frac{t^2}{2(2^n-1)}}\, dt}_{\ge 0}\Big]$$

Then, $T_{x,y} \ge \frac{4}{\sqrt{2\pi \cdot 2^n}} \cdot \int_0^D P\big(t < f(x)\big) \cdot e^{-\frac{t^2}{2(2^n-1)}}\, dt$

Again, using that the function $e^{-\frac{t^2}{2(2^n-1)}}$ is strictly decreasing on the interval $(0, \infty)$, we obtain that:

$T_{x,y} \ge \frac{4}{\sqrt{2\pi \cdot 2^n}} \cdot e^{-\frac{D^2}{2(2^n-1)}} \cdot \int_0^D P\big(t < f(x)\big)\, dt = \frac{4}{\sqrt{2\pi \cdot 2^n}} \cdot e^{-\frac{D^2}{2(2^n-1)}} \cdot \left(\int_0^{\infty} P\big(t < f(x)\big)\, dt - \int_D^{\infty} P\big(t < f(x)\big)\, dt\right)$

$$T_{x,y} \ge \frac{4}{\sqrt{2\pi \cdot 2^n}} \cdot e^{-\frac{D^2}{2(2^n-1)}} \cdot \left(\int_0^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{t^2}{2}} \cdot t\, dt - \int_D^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{t^2}{2}} \cdot t\, dt\right)$$

$$T_{x,y} \ge \frac{4}{\sqrt{2\pi \cdot 2^n}} \cdot e^{-\frac{D^2}{2(2^n-1)}} \cdot \left(\frac{1}{\sqrt{2\pi}} - \frac{e^{-\frac{D^2}{2}}}{\sqrt{2\pi}}\right) = \frac{2}{\pi \cdot \sqrt{2^n}} \cdot e^{-\frac{D^2}{2(2^n-1)}} \cdot \left(1 - e^{-\frac{D^2}{2}}\right)$$

If we choose $D = \sqrt{2n}$ we will obtain:

$T_{x,y} \ge \frac{2}{\pi \cdot \sqrt{2^n}} \cdot e^{-\frac{n}{2^n - 1}} \cdot (1 - e^{-n}) \ge \frac{2}{\pi\sqrt{2^n}} - O\left(\frac{n}{2^{\frac{3n}{2}}}\right) \qquad \forall x, y \in \{0,1\}^n$

$\Theta_{F,G}$ has $2^n \cdot 2^n$ terms $T_{x,y}$, therefore we reach that:

$$\mathrm{E}(\Theta_{F,G}) \le \frac{1}{2^{\frac{3n}{2}}} \cdot 2^n \cdot 2^n \cdot \left(\frac{2}{\pi\sqrt{2^n}} + O\left(\frac{1}{2^{\frac{3n}{2}}}\right)\right) = \frac{2}{\pi} + O\left(\frac{1}{2^n}\right) \qquad (1)$$

$$E\left(\Theta_{F,G}\right) \geq \frac{1}{2^{\frac{3n}{2}}} \cdot 2^n \cdot 2^n \cdot \left(\frac{2}{\pi\sqrt{2^n}} - O\left(\frac{n}{2^{\frac{3n}{2}}}\right)\right) = \frac{2}{\pi} - O\left(\frac{n}{2^n}\right) \quad (2)$$

From (1) and (2) we get $E\left(\Theta_{F,G}\right) \cong \frac{2}{\pi} \geq \frac{3}{5}$ [32], which is what we wanted to show.

The second part of the reduction consists of proving that if every $f(x)$ and $g(y)$ are uncorrelated, then $|\Theta_{F,G}| \leq \frac{1}{100}$.

In this case, $T_{x,y} = E[(-1)^{x \cdot y} \cdot F(x) \cdot G(y)] = 0 \; \forall x, y \in \in \{0,1\}^n, x \neq y$ (as now $F(x)$ and $G(y)$ are also independent [34])

Then, in the $\Theta_{F,G}$ sum we are left with only 2^n terms, so we obtain:

$E[\Theta_{F,G}] = \frac{1}{2^{\frac{3n}{2}}} \cdot 2^n = \frac{1}{2^{\frac{n}{2}}} < \frac{1}{100}$.

Consequently, we have shown that there **exists a reduction** from the ***Real Forrelation*** problem to ***Forrelation***. □

Because we need to relate *Real Forrelation* to the k*-Forrelation* problem (only the k*-Forrelation* problem is $BQP - complete$ when $k = O(n)$), we also build a version of the *Real Forrelation* problem involving k functions.

DEFINITION 12. In the k*-Real Forrelation* [25] we have a sequence of k functions $f_1, f_2, \dots, f_k$, where $f_1, f_2, \dots, f_{k-2} : \{0,1\}^n \to \{-1,1\}$ and $f_{k-1}, f_k : \{0,1\}^n \to \mathbb{R}$. We know that all values $f_i(x) \; \forall i \in \{1, \dots, k-2\}$ are independently drawn from the set $\{-1,1\}$. Moreover, we are promised that for f_{k-1}, f_k, one of the following 2 cases hold:

i) Every $f_{k-1}(x)$ and $f_k(y)$ are both drawn from 2 normal distributions $\forall x, y \in \{0,1\}^n$
ii) Every $f_{k-1}(x)$ is drawn from a normal distribution, but the function f_k depends on the values of all other $k-1$ functions:

$$f_k(x) = \frac{1}{2^{\frac{(k-1)n}{2}}} \sum_{x_1, x_2, \dots, x_{k-1} \in \{0,1\}^n} (-1)^{x_1 \cdot x_2} \cdot \dots \cdot (-1)^{x_{k-1} \cdot x} \cdot f_1(x_1) \cdot \dots \cdot f_{k-1}(x_{k-1})$$

As before, the problem asks to determine which of the cases applies given a sequence of k functions.

THEOREM 3.2.2. k*-Real Forrelation* is *equivalent* to the problem *Real Forrelation* [25].

Proof. Consider $(f_1, f_2, \dots, f_k)$ an instance of the k*-Real Forrelation*. Then, we can build 2 functions $f, g : \{0,1\}^n \to \{-1,1\}$ in the following way:

$g(x) = f_k(x)$

$f(x) = f_{k-1}(x) \cdot d_x$, with

$$d_x = \frac{1}{2^{\frac{(k-2)n}{2}}} \sum_{x_1 \dots x_{k-2} \in \{0,1\}^n} (-1)^{x_1 \cdot x_2} \cdot \dots \cdot (-1)^{x_{k-3} \cdot x_{k-2}} \cdot f_1(x_1) \cdot \dots \cdot f_{k-2}(x_{k-2}) \cdot (-1)^{x_{k-2} \cdot x}$$

We make the first observation that: $\Theta_{f,g} = \Theta_{f_1,f_2,\dots,f_k}$. Indeed, if we replace f and g with their definitions in $\Theta_{f,g}$, we get:

$\Theta_{f,g} = \frac{1}{2^{\frac{3n}{2}}} \sum_{x,y \in \{0,1\}^n} (-1)^{x \cdot y}\, g(y) \cdot f(x) = \frac{1}{2^{\frac{3n}{2}}} \sum_{x,y \in \{0,1\}^n} (-1)^{x \cdot y}\, f_k(y) \cdot f_{k-1}(x) \cdot\ d_x =$

$= \frac{1}{2^{\frac{3n}{2}}} \sum_{x,y \in \{0,1\}^n} (-1)^{x \cdot y}\, f_k(y) \cdot f_{k-1}(x) \cdot \left[\frac{1}{2^{\frac{(k-2)n}{2}}} \sum_{x_1 \dots x_{k-2} \in \{0,1\}^n} (-1)^{x_1 \cdot x_2} \cdot \dots \cdot (-1)^{x_{k-3} \cdot x_{k-2}} \cdot \right.$
$\left. f_1(x_1) \cdot \dots \cdot f_{k-2}(x_{k-2}) \cdot (-1)^{x_{k-2} \cdot x} \right] =$

$= \frac{1}{2^{\frac{(k+1)n}{2}}} \sum_{x,y,x_1 \dots x_{k-2} \in \{0,1\}^n} (-1)^{x_1 \cdot x_2} \dots (-1)^{x_{k-2} \cdot x} (-1)^{x \cdot y}\, f_k(y) \cdot f_{k-1}(x) \cdot f_1(x_1) \cdot \dots \cdot f_{k-2}(x_{k-2})$
$= \Theta_{f_1,f_2,\dots,f_k}$

Now if $f_{k-1}(x)$ and $f_k(x)$ are both uncorrelated normal distributions, then because $g = f_k$ and f is a linear combination of values of the functions $f_1, f_2, \dots, f_{k-2}$, we obtain that every $f(x)$ and $g(x)$ values are uncorrelated normal distributions [28, 34].

What is left to show is that if the functions $\{f_1(x), \dots f_{k-1}(x), f_k(x)\}$ are in the second case of *k-Real Forrelation*, then f and g are also in the second scenario of *Real Forrelation*.

We have $g(x) = f_k(x) = \frac{1}{2^{\frac{(k-1)n}{2}}} \sum_{x_1,\dots,x_{k-1} \in \{0,1\}^n} (-1)^{x_1 \cdot x_2} \cdot \dots \cdot (-1)^{x_{k-1} \cdot x} \cdot f_1(x_1) \cdot \dots \cdot\ f_{k-1}(x_{k-1})$

$g(x) = \sum_{x_{k-1} \in \{0,1\}^n} f_{k-1}(x_{k-1}) \cdot \frac{1}{2^{\frac{(k-1)n}{2}}} \sum_{x_1 \dots x_{k-2} \in \{0,1\}^n} (-1)^{x_1 \cdot x_2} \cdot \dots \cdot (-1)^{x_{k-1} \cdot x} \cdot f_1(x_1) \cdot \dots \cdot$
$f_{k-2}(x_{k-2}) = \frac{1}{\sqrt{2^n}} \sum_{x_{k-1} \in \{0,1\}^n} f_{k-1}(x_{k-1}) \cdot\ d_{x_{k-1}} \cdot (-1)^{x_{k-1} \cdot x}$

$g(x) = \frac{1}{\sqrt{2^n}} \sum_{x_{k-1} \in \{0,1\}^n} f(x_{k-1}) \cdot (-1)^{x_{k-1} \cdot x}$

Therefore, we have proved that the *k-Real Forrelation* problem is equivalent to the problem *Real Forrelation* (with only 2 functions). ☐

Now, we can finally show that ***k-Real Forrelation*** can be **reduced to *k-Forrelation***.

Proof. Given an instance of *k-Real Forrelation*, $f_1, f_2, \dots, f_{k-2} : \{0,1\}^n \to \{-1,1\}$ and $f_{k-1}, f_k :$ $\{0,1\}^n \to \mathbb{R}$, we build an instance of *k-Forrelation* $F_1, F_2, \dots, F_k : \{0,1\}^n \to \{-1,1\}$ in this way:

$F_1(x) = f_1(x), \dots, F_{k-2}(x) = f_{k-2}(x)$

$F_{k-1}(x) = sgn(f_{k-1}(x)), F_k(x) = sgn(f_k(x))$

Then, $\Theta_{F_1,F_2,\dots,F_k} = \frac{1}{2^{\frac{(k+1)n}{2}}} \cdot \sum_{x_1,x_2,\dots,x_k \in \{0,1\}^n} (-1)^{x_1 \cdot x_2} \cdot \dots \cdot (-1)^{x_{k-1} \cdot x_k} \cdot\ f_1(x_1) \cdot \dots \cdot f_{k-2}(x_{k-1}) \cdot$
$F_{k-1}(x_{k-1}) \cdot F_k(x_k) = \frac{1}{2^{\frac{3n}{2}}} \sum_{x_{k-1},x_k \in \{0,1\}^n} (-1)^{x_{k-1} \cdot x_k} \cdot F_{k-1}(x_{k-1}) \cdot F_k(x_k) \cdot d_{x_{k-1}}$

Next, we can take the expected value of each term of this sum, $E(F_{k-1}(x) \cdot F_k(y) \cdot d_x)$ and show as we did for the *Real Forrelation – Forrelation* reduction that $E(\Theta_{F_1,F_2,\dots,F_k}) \cong \frac{2}{\pi} \geq \frac{3}{5}$ when f_k depends on the values of $f_1, f_2, \dots, f_{k-1}$ and $E(\Theta_{F_1,F_2,\dots,F_k}) \cong 0 \leq \frac{1}{100}$ [30, 32] when f_k values are uncorrelated.

Therefore, we have obtained that *k-Forrelation* is harder to solve than *k-Real Forrelation*, which in turn is equivalent to *Real Forrelation*. ☐

At this point, the remaining part is to show that *Real Forrelation* cannot be solved by any machine in $NP/poly^O$.

8.3.1. Distribution Correlation

Instead of working with *Real Forrelation*, we consider an abstraction of this problem, called *Distribution Correlation* [25].

DEFINITION 13. In *Distribution Correlation* we have a set of 2^{n+1} vectors, $\mathcal{W} = \{w_1, w_2, \dots, w_{2^{n+1}}\}$, where $w_i \in \mathbb{R}^{2^n}$. This problem is a "black box" model [23], for each vector w_i queried we receive in response a random number c_i, with the promise that either:

i) all responses for all vectors $\{c_1, c_2, \dots, c_{2^{n+1}}\}$ are independently chosen from a normal distribution *or*

ii) For every response, we have that: $c_i = (w_i, v)$ - the inner vector between w_i and v, where v is a fixed vector from $\mathbb{R}^{2^{n+1}}$.

The problem asks to determine if $\{c_1, c_2, \dots, c_{2N}\}$ are chosen from the normal distribution (case i) or if there exists a *positive covariance* [28] between them (case ii).

The complexity reduces to the number of queries we have to make in order to decide in which of the 2 situations we are.

OBS **10.** We notice that we can solve this problem deterministically using at most $2^n + 1$ queries. We can view *Distribution Correlation* as the problem of deciding if there exists a vector $v \in \mathbb{R}^{2^n}$, such that for any response c_i, we have that $c_i = (w_i, v)$. Suppose there exists such a vector v.

Then, for each query we make, we get a new linear equation involving the vector v:

$$\begin{cases} (w_1, v) &= c_1 \\ (w_2, v) &= c_2 \\ \quad \dots & \\ (w_{2^n}, v) &= c_{2^n} \end{cases}$$

Consequently, after 2^n queries we obtain a system of 2^n linear equations containing the 2^n elements of v. Therefore, we can solve the resulting linear system and obtain v. To determine in which of the 2 cases of the problem we are, all we need to do is to make one more query for the vector w_{2^n+1}. Then, we check if the response we receive from the oracle, c_{2^n+1}, verifies the relation: $c_{2^n+1} = (w_{2^n+1}, v)$ (with the v computed before). If it does, this implies that there actually exists a positive covariance between the oracle's responses, so we must be in the second case. Otherwise, it means that *not* all query responses c_i verify the property that there exists a vector v such that: $c_i = (w_i, v)$, so we are in the former situation.

OBS **10.** Analysing the *Distribution Correlation* problem, we deduce it is similar to *Simon's* problem. *Simon's* problem can be solved if we show the existence of the *bit-mask* string s, while the *Distribution Correlation* problem can be decided if we prove the existence of the vector v.

Furthermore, in *Simon's* problem we queried the oracle for the input x_i to obtain the value of the function in that point, $f(x_i)$. Equivalently in the current problem we query w_i to obtain the corresponding random value c_i.

Both *Simon* and *Distribution Correlation* can only be solved by *Oracle Turing Machines* [10].

Thus, any machine M solving the *Distribution Correlation* problem would have in its computational tree at each node a query w_i to the oracle. Then, based on all the query answers $(c_1, \dots, c_l)$ along a path in its computational tree, M will decide whether all c_i's are independent or not.

Considering all these arguments, we can follow the same proofs we used for MAIN THEOREM 2.2 and MAIN THEOREM 2.3, in order to show the two results below.

COROLLARY 2: *Distribution Correlation* $\notin NP^O$.

Proof Sketch. The oracle receives the dimensions of our vectors, 2^n, and the vector with the associated index i, and returns c_i. We can show that *Distribution Correlation* cannot be solved by a nondeterministic machine using *diagonalization* [21]. For each nondeterministic $TM\ M_i$, the oracle O finds a vector of responses $\{c_1, \dots, c_{2^n}\}$ which is accepted by M_i. Because the *adversarial oracle* knows the computational tree of M_i he can trick M_i in giving the wrong decision [31]. M_i accepts an input I is equivalent to saying that there exists a path π in his computational tree containing a polynomial number of queries $\{w_{j_1}, w_{j_2}, \dots, w_{j_k}\}$ (with their associated answers $\{c_{j_1}, c_{j_2}, \dots, c_{j_k}\}$), $such\ that\ \pi$ accept I. Then, the oracle O can keep the answers to these k queries, but can still set the values for the rest of the answers c_i such that there exists a vector v with the property that $< w_j, v > = c_j$ $\forall i \in \{1, \dots, 2^n\}$. By fixing the answers $\{c_{j_1}, c_{j_2}, \dots, c_{j_k}\}$ we just set k out of 2^n elements of v, so no matter what M_i queries on path π, the oracle can deceive him to accept the input. $\square$

COROLLARY 3: *Distribution Correlation* $\notin NP/poly^O$.

Proof Sketch. For the $NP/poly$ case we take into consideration every polynomial-size advice a nondeterministic machine M_i can receive [32, 33]. For each of them we basically have a new nondeterministic machine which could solve *Distribution Correlation*. Our aim is to show that we can eliminate all these machines by properly setting a different set of answers $\{c_1, \dots, c_{2^n}\}$ for each input index. Because M_i receives the same advice for all sets of vectors from $\mathbb{R}^{2^n}$, for every input to M_i, we can eliminate a fraction of the total number of these advices the same way we did for the *coSimon problem* [23]. $\square$

In this section we have presented another candidate problem which suggests the oracle separation between BQP and $NP/poly$. We have started from a $BQP - complete$ problem, k*-Forrelation*, and through a sequence of equivalences and reductions between problems, we have reached the *Distribution Correlation* problem. By observing a series of similarities between *Distribution Correlation* and *Simon's problem*, we applied similar arguments to indicate that *Distribution Correlation* cannot belong to the class $NP/poly^O$.

Because the chain of relations described in THEOREM 3.2.1 and THEOREM 3.2.2 tell us that *Distribution Correlation* is at most as "difficult" as the $BQP - complete$ k*-Forrelation* problem, this would imply that **the no-go result** also **holds for *k-Forrelation*.**

Consequently, because $BQP - hard$ represents the most *difficult* problems from BQP [19], we infer that for any problem $\mathcal{P}$ in $BQP - hard$ there is no *oracle* $NP/poly$ machine that can solve $\mathcal{P}$.

9. Conclusions

Quantum technologies are evolving at a rapid pace motivated by the staggering potential of quantum computers to reach a computational efficiency unattainable by any classical machine [3].

The prospect of such a powerful computer puts us in the position to address the following vital problem: To what extent we can take advantage of the computational resources of a quantum computer in a way that our data is protected?

In this paper we have analysed the encryption scheme *GES* which allows a classical client to delegate an unfeasible computation to a quantum server and derive the result of the computation such that his input remains private from the server.

We illustrated the usefulness of the *GES* framework by defining encryption schemes which solve efficiently some hard problems (*discrete logarithm*, *primitive root*, *quadratic residuosity*) [16].

Then, we focused on the main theme of the project: the relation between the difficulty of encrypting a function using a *GES* protocol and the difficulty of solving it. We studied this problem from a *complexity theoretic* point of view. Firstly, we reviewed the recent work of Abadi who proved that there is a connection between encryptability and nonuniform complexity [16, 24].

Based on this result, we investigated if the class of problems efficiently solvable by a quantum computer (BQP problems) admits a *GES* protocol. We conjecture that not all BQP problems can be solved in the *GES* scenario. This would be equivalent to showing an outstanding complexity theory result: BQP is not included in the nonuniform class $NP/poly$. However, proving that BQP is not contained in $NP/poly$ is at least as difficult as showing that P is not contained in NP [17, 20]. Consequently, we decided to give strong arguments in favour of the separation between the 2 classes by showing that BQP cannot be included in $NP/poly$ with respect to an oracle (we *relativized* [31] the relation between the quantum and the classical complexity class).

To prove the oracle separation between BQP and $NP/poly$ we considered two candidates: *Simon's problem* and *Forrelation*. We first proved that the two problems can be solved using BQP algorithms. Then, using an innovative method akin to an exhaustive search, we showed that we can define an oracle O such that no possible polynomial advice could help an NP machine solve these problems. Consequently, they cannot belong to the class $NP/poly^O$. Finally, because all problems from $BQP - Hard$ are more difficult than any problem from BQP (including *Simon* and *Forrelation*), this implies that none of these problems can be solved by an $NP/poly^O$ algorithm.

As a result, we presented powerful evidence that $BQP - Hard$ problems cannot be solved in this secure delegated computation task.

Our second contribution was related to the $UBQC$ quantum protocol [26] which allows a client possessing some minimal quantum abilities, to determine the result of a hard computation by communicating with a quantum computer, while keeping the input and the computation itself private.

Inspired by the idea of having a purely classical client who could benefit from the security advantages guaranteed by *UBQC*, we investigated the existence of a classical version of *UBQC* (*CUBQC*). We proved that this resulting scheme could be represented as an instance of the *GES*. Therefore, *CUBQC* inherits all the complexity theoretic results known for *GES*.

All these results achieved in our project inspire us to pose the following open problems:

- Do the no-go theorems regarding the encryptability of $NP-Hard$ and $BQP-Hard$ functions still hold if we modify the security conditions of the *GES* framework (for instance allowing the server to infer more than just the size of the input)?
- Consider a *quantum generalization* of the *GES* scheme, *QGES*, where the client has some limited quantum abilities, but still requires the superior computational abilities of a quantum server. Can we derive quantum analogue complexity theory results for *QGES*; namely, can we derive that in the *QGES* scenario any BQP function can be encrypted without leaking anything and that any encryptable function can be solved by a $QNP/poly$ algorithm?

In conclusion, we accomplished the goals we set for this project and the no-go results we indicated lead to many new questions which must be dealt with, in order to fully understand what are the limitations imposed by a secure exploit of the high performance of a quantum computer.

Bibliography

[1] Michael A. Nielsen, Isaac L. Chuang. 2010. Quantum Computation and Quantum Information. Cambridge University Press.

[2] Simon Bone, Matias Castro. 1997. A Brief History of Quantum Computing. Surveys and Information Systems Engineering, Volume 4.

[3] Jacob West. 2000. The Quantum Computer. Rice University, Houston.

[4] Charles Bennett, Ethan Bernstein, Gilles Brassard, Umesh Vazirani. 1997. Strengths and Weaknesses of Quantum Computing. SIAM Journal of Computing, Vol. 26, pp 1510-1523.

[5] Dorit Aharonov, Umesh Vazirani. 2012. Is Quantum Mechanics Falsifiable? A computational perspective on the foundations of Quantum Mechanics. Philosophy of Science anthology, MIT press.

[6] David Deutsch. 1985. Quantum theory, the Church-Turing principle and the universal quantum computer. Proceedings of the Royal Society of London A 400, pp 97-117.

[7] Peter Shor. 1996. Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer. Proceedings of the 35th Annual Symposium on Foundations of Computer Science.

[8] Ignacio Cirac, Peter Zoller. 2012. Goals and opportunities in quantum simulation. Nature Physics, Volume 8, Issue 4, pp 264-266.

[9] Reihaneh Safavi-Naini. 2008. Information Theoretic Security. Proceedings of the 3rd International Conference ICITS Calgary.

[10] Ethan Bernstein, Umesh Vazirani. 1997. Quantum Complexity Theory. SIAM Journal of Computing, Volume 26, Issue 5, pp 1411-1473.

[11] G. Brassard, Bennett. 1984. Quantum cryptography: Public key distribution and coin tossing. Proceedings of IEEE International Conference on Computers, Systems, vol. 175.

[12] Tim Moses. 2009. Quantum Computing and Cryptography. ETSI White Papers No. 8.

[13] Scott Aaronson, Andrew Drucker. 2011. Advice Coins for Classical and Quantum Computation. Proceedings of International Colloquium on Automata, Languages and Programming, pp 61-72.

[14] Andrew M. Childs. 2005. Secure assisted quantum computation. Journal Quantum Information & Computation, Vol. 5, Issue 6, pp 456-466.

[15] S. Barz, E. Kashefi, A. Broadbent, J. Fitzsimons. 2012. Demonstration of Blind Quantum Computing. Science, Vol. 335, pp. 303-308.

[16] Martin Abadi, Joan Feigenbaum, Joe Kilian. 1989. On hiding Information from an oracle. Journal of Computer and System Sciences, Volume 39, Issue 1, pp 21-50.

[17] Sanjeev Arora, Boaz Barak. 2009. Computational Complexity: A Modern Approach. Cambridge University Press.

[18] Neal Koblitz, Alfred Menezes. 2013. Another look at nonuniformity. Groups Complexity Cryptology, Volume 5, pp 117-139.

[19] Scott Aaronson. 2010. BQP and the Polynomial Hierarchy. Proceedings of the 42nd Annual ACM Symposium on Theory of Computing, pp 141-150.

[20] Jose L. Balcazar, Josep Diaz, Joaquim Gabarro. 1988. Structural Complexity I. Springer-Verlag.

[21] Jose L. Balcazar, Josep Diaz, Joaquim Gabarro. 1990. Structural Complexity II. Springer-Verlag.

[22] Stuart Kurtz. 1983. On the Random Oracle Hypothesis. Proceedings of 14th annual ACM Symposium on Theory of Computing, pp 224-230.

[23] Daniel R. Simon. 1997. On the power of Quantum Computation. Journal on Computing, Volume 26, Issue 5, pp 1474-1483.

[24] Johannes Kobler, Thomas Thierauf. 1990. Complexity Classes with Advice. Proceedings of the 5th IEEE Conference Structure in Complexity Theory, pp 305-315.

[25] Scott Aaronson, Andris Ambainis. 2014. Forrelation: A problem that Optimally Separates Quantum from Classical Computing. Proceedings of the 47th Annual ACM Symposium on Theory of Computing, pp 307-316.

[26] Anne Broadbent, Joseph Fitzsimons, Elham Kashefi. 2009. Universal blind quantum computation. Proceedings of the 50th Annual IEEE Symposium on Foundations of Computer Science 2009, pp 517-526.

[27] Stefanie Barz, Vedran Dunjko, Florian Scglederer, Merrit Moore, Elham Kashefi, Ian Walmsey. 2015. Enhanced delegated computation using coherence. Physical Review A.

[28] Felix Weninger. 2006. Randomness and non-uniformity. Joint Advanced Student School.

[29] Petros Wallden. 2015. Introduction to Quantum Computing. University of Edinburgh

[30] Scott Joel Aaronson. 2000. Limits on Efficient Computation in the Physical World. Ph.D. thesis University of California, Berkeley.

[31] Frederic Green, Randall Pruim. 2000. Relativized separation of EQP from P^{NP}. Information Processing Letters, Volume 80, Issue 5, pp 257-260.

[32] Ragesh Jaiswal. 2008. New Proofs of (New) Direct Product Theorems. Ph.D. thesis University of California, San Diego.

[33] Stephen A. Cook. 1971. The Complexity of Theorem-Proving Procedures. Proceedings of the 3rd ACM Symposium on Theory of Computing, pp 151-158.

[34] Paul Hoel, Sidney Port, Charles Stone. 1971. Introduction to Probability Theory. Cengage Learning.

[35] Lawrence Carter, Mark Wegman. 1979. Universal classes of hash functions. Journal of Computer and System Sciences, Volume 18, Issue 2, pp 143-154.